To show that $\lim_{d\to\infty} \dfrac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, we need to take the problem in two parts. Recall that the limit of a product is the product of the limit, providing the two limits exist. So, we will consider two limit problems. First, consider the functional form $\lim_{d\to\infty} \dfrac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}}$ and then address the value of the constant $\lim_{d\to\infty} \dfrac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!}$.

$$\lim_{d\to\infty} \frac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} \text{ and } \lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!}.$$

The Functional Form $\lim_{d\to\infty} \dfrac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} = e^{-\frac{1}{2}x^2}$

This first limit is easier than the second, so we begin by considering the function in the denominator, $\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}$. The $\lim_{d\to\infty}\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}$ is of the indeterminate form 1^{∞}. So, we proceed in the standard manner. Consider $y=\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}$, so $\ln y = \left(\frac{d+1}{2}\right)\ln\left(1+\frac{x^2}{d}\right)$. This expression is in the indeterminate form $\infty\cdot 0$, so we rewrite to find $\ln y = \dfrac{\ln\left(1+\frac{x^2}{d}\right)}{\left(\frac{2}{d+1}\right)}$, which has indeterminate form $\frac{0}{0}$. Now, apply L'Hopital's rule.

We consider $\lim_{d\to\infty}\dfrac{\left(\dfrac{\left(\dfrac{-x^2}{d^2}\right)}{\left(1+\dfrac{x^2}{d}\right)}\right)}{\left(\dfrac{-2}{(d+1)^2}\right)}=\lim_{d\to\infty}\left(\dfrac{-x^2}{d^2}\right)\left(\dfrac{1}{1+\dfrac{x^2}{d}}\right)\left(\dfrac{(d+1)^2}{-2}\right)=\dfrac{x^2}{2}$.

Then $\lim_{d\to\infty}\ln y=\dfrac{x^2}{2}$ by L'Hopital's rule and $\lim_{d\to\infty} y=e^{\frac{x^2}{2}}$.

Finally, $\lim_{d\to\infty}\dfrac{1}{\left(1+\dfrac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}}=\dfrac{1}{e^{\frac{x^2}{2}}}=e^{-\frac{x^2}{2}}$ which we recognize as the functional structure of the Normal distribution.

Now, to show that $\lim_{d\to\infty}\dfrac{\left(\dfrac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\dfrac{d-2}{2}\right)!}=\dfrac{1}{\sqrt{2\pi}}$.

We begin by pulling out the constant $\dfrac{1}{\sqrt{\pi}}$ to simplify the work. So,

$$\lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!}=\left(\frac{1}{\sqrt{\pi}}\right)\lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d}\left(\frac{d-2}{2}\right)!}$$

and we need to show that $\lim_{d\to\infty}\dfrac{\left(\dfrac{d-1}{2}\right)!}{\sqrt{d}\left(\dfrac{d-2}{2}\right)!}=\dfrac{1}{\sqrt{2}}$.

If we rewrite using Stirling's formula and separate the limit of the product into the product of three limits, then

$$\lim_{d\to\infty}\frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}\,e^{\left(\frac{d-1}{2}\right)}\left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\,\sqrt{2\pi\left(\frac{d-2}{2}\right)}\,e^{\left(\frac{d-2}{2}\right)}\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}$$

$$=\left(\lim_{d\to\infty}\frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}}{\sqrt{2\pi\left(\frac{d-2}{2}\right)}}\right)\left(\lim_{d\to\infty}\frac{e^{\left(\frac{d-1}{2}\right)}}{e^{\left(\frac{d-2}{2}\right)}}\right)\left(\lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}\right).$$

We can take each of the three limits one at a time. The first is quite straightforward,

$$\lim_{d\to\infty}\frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}}{\sqrt{2\pi\left(\frac{d-2}{2}\right)}}=1.$$

The second requires a bit of rewriting,

$$\lim_{d\to\infty}\frac{e^{-\left(\frac{d-1}{2}\right)}}{e^{-\left(\frac{d-2}{2}\right)}}=\lim_{d\to\infty}\frac{e^{-\left(\frac{d}{2}\right)}e^{\frac{1}{2}}}{e^{-\left(\frac{d}{2}\right)}e^{1}}=e^{-\frac{1}{2}}.$$

The last term requires the most work. We rewrite

$$\lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}=\lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)^{\left(\frac{d-2}{2}\right)}\sqrt{\left(\frac{d-1}{2}\right)}}{\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}\sqrt{d}}=\lim_{d\to\infty}\left(\frac{d-1}{d-2}\right)^{\left(\frac{d-2}{2}\right)}\lim_{d\to\infty}\sqrt{\left(\frac{d-1}{2d}\right)}.$$

The first term is indeterminate of the form 1^{∞}, so it can be evaluated using L'Hopital's Rule.

Let $y=\left(\frac{d-1}{d-2}\right)^{\left(\frac{d-2}{2}\right)}$, so $\ln y=\dfrac{\ln\left(\frac{d-1}{d-2}\right)}{\left(\frac{2}{d-2}\right)}$.

Consider, $\lim_{d\to\infty} \frac{\left(\frac{d-2}{d-1}\right)\left(\frac{(d-1)-(d-2)}{(d-2)^2}\right)}{\left(\frac{-2}{(d-2)^2}\right)} = \lim_{d\to\infty} \frac{1}{2}\left(\frac{d-2}{d-1}\right) = \frac{1}{2}$. So, $\lim_{d\to\infty} y = e^{\frac{1}{2}}$.

Finally, we have $\lim_{d\to\infty} \sqrt{\left(\frac{d-1}{2d}\right)} = \frac{1}{\sqrt{2}}$. Putting it all together, we have

$$\lim_{d\to\infty} \frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}\, e^{\left(\frac{d-1}{2}\right)} \left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\, \sqrt{2\pi\left(\frac{d-2}{2}\right)}\, e^{\left(\frac{d-2}{2}\right)} \left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}$$

$$= \left(\lim_{d\to\infty} \frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}}{\sqrt{2\pi\left(\frac{d-2}{2}\right)}}\right)\left(\lim_{d\to\infty} \frac{e^{\left(\frac{d-1}{2}\right)}}{e^{\left(\frac{d-2}{2}\right)}}\right)\left(\lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}\right)$$

$$= (1)\left(e^{-\frac{1}{2}}\right)\left(e^{\frac{1}{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

This was the desired result. So,

$$\lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\,\left(\frac{d-2}{2}\right)!\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

and the *t*-distribution approaches the standard normal distribution as the degrees of freedom increase without bound.