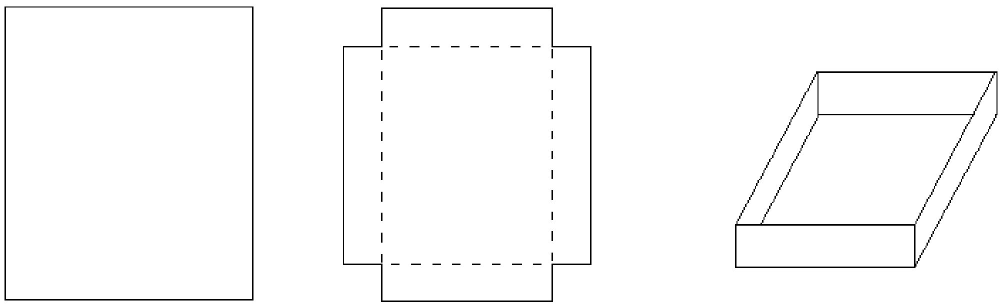
Calculus Challenge Problem #5 **Solution**

A New Look at an Old Problem

(A special thanks to Scott Farrand, Professor of Mathematics at the California State University at Sacramento for this problem.)

By now, you will likely have considered "the box problem". In this problem, you begin with a rectangular piece of paper, cut out squares in the corner, and make a box with the largest possible volume.



1) Find the dimensions of the largest box that can be made from a sheet of paper that is 10 inches wide and 16 inches long.

a) For this box, compare the area of the base to the area of the sides of this box.

This is the classic problem of elementary calculus. We have $V(x) = x(10-2x)(16-2x)$ with the domain of x being $[0,5]$. We know what this function looks like and this will make things easier later. We have a cubic function with a positive lead coefficient and zeros at 0, 5, and 8. It must have a shape like that shown at right.

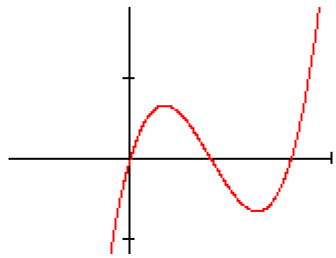
So, $V(x) = 160x - 52x^2 + 4x^3$ and $V'(x) = 160 - 104x + 12x^2$. This quadratic has zeros at $x = 2$ and at $x = \frac{20}{3}$. Only $x = 2$ is in the domain. Moreover, $V''(x) = -104 + 24x$, so $V''(2) < 0$. The maximum volume is 144 cubic inches when squares 2 inches on a side are removed.

The area of the base is, $A_B(x) = (10-2x)(16-2x)$, is 72 square inches when $x = 2$. The area of the sides is, $A_S(x) = 2[x(10-2x)] + 2[x(16-2x)]$, is also 72 square inches when $x = 2$. The largest box is formed when the area of the sides is the same as the area of the base.

Is this always true? Will the maximum volume occur when the two areas are the same, or was there something special about this box (after all, how many of these boxes have integer measurements like this one)?

b) Can you find another rectangle that has this same relationship between the area of the base and the area of the sides for the largest volume? Is the relationship you see a general one for all boxes made from rectangles or is there something special about an 10 x 16 inch sheet of paper that produces this result?

Repeat the analysis above, but with the general equation $V(x) = x(W-2x)(L-2x)$ with the domain of x being $\left[0, \min\left(\frac{W}{2}, \frac{L}{2}\right)\right]$. The convention is to consider the width the shorter side, so $x \in \left[0, \frac{W}{2}\right]$. Again, we know the shape of this cubic curve with zeros at 0, $\frac{W}{2}$, and $\frac{L}{2}$ which increases without bound as x increases. When we solve the quadratic below, we know the maximum is at the smaller root. Now, $V(x) = WLx - 2(W+L)x^2 + 4x^3$, so $V'(x) = WL - 4(W+L)x + 12x^2$.



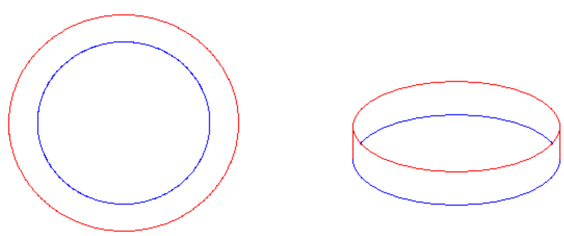
The roots are $x = \dfrac{(W+L) - \sqrt{W^2 - WL + L^2}}{6}$.

Rather than go through the messy process of evaluating the area functions for this x, we will take a different approach to try to make things a little neater and perhaps easier.

We want to know if $V'(x) = WL - 4(W+L)x + 12x^2 = 0$ gives us equal areas for the base and the sides. The area of the base is $A_B(x) = (W-2x)(L-2x)$ and the area of the sides $A_S(x) = 2x(W-2x) + 2x(L-2x)$.

If $WL - 4(W+L)x + 12x^2 = 0$ then $WL - 2(W+L)x + 4x^2 + \left(-2x(W+L) + 8x^2\right) = 0$ so $WL - 2(W+L)x + 4x^2 = 2x(W+L) - 8x^2$. But this is $(W-2x)(L-2x) = 2x(W-2x) + 2x(L-2x)$, so the areas are always equal.

2) Suppose we use a round piece of paper to make a circular "box" by folding up the sides. Of course, the sides will crinkle up, but imagine we can cut out the excess and throw it away. What is the relationship between the area of the base and the area of the side of this "box".



For the circle, we have an external radius of R, a ring of width h, and an internal radius of r. With, $h = R - r$ and $V = \pi r^2 h = \pi r^2 (R - r)$, we have $V(r) = \pi R r^2 - \pi r^3$.

$V'(r) = 2\pi R r - 3\pi r^2$. If $V'(r) = 0$, then $r = 0$ and $r = \dfrac{2R}{3}$.

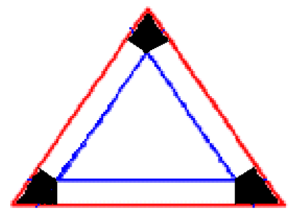
$V''(r) = 2\pi R - 6\pi r$, so $V''\left(\frac{2R}{3}\right) < 0$.

What about the areas of the base and the sides?

$A_B = \pi r^2 = \pi\left(\frac{2R}{3}\right)^2 = \frac{4\pi R^2}{9}$ and $A_S = 2\pi rh = 2\pi\left(\frac{2R}{3}\right)\left(\frac{R}{3}\right) = \frac{4\pi R^2}{9}$. Again, they are the same.

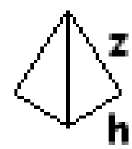
Finally, we have the triangle.

3) Try this with an equilateral triangle. Cut out kites in the corners and fold up the sides to make the box of maximum volume. How does the area of the sides compare to the area of the base? Show your work.



This part has the most demanding geometry.

If the length of a side is S and h is the length of the perpendicular to the side and the height of the box, then the kite has two right angles, an obtuse angle of 120 degrees and an acute angle of 60 degrees. Half the kite is a 30-60-90 triangle, so we know that the length of z is $\sqrt{3}h$. This means the length of a side of the base is $S - 2\sqrt{3}h$.



The Volume of the box is $V = h\left(A_{Base}\right) = \frac{\sqrt{3}}{4}\left(S - 2\sqrt{3}h\right)^2$ since the base is an equilateral triangle with side length $S - 2\sqrt{3}h$. If $V = 3\sqrt{3}h^3 - 3Sh^2 + \frac{\sqrt{3}}{4}S^2 h$, then $\frac{dV}{dh} = 9\sqrt{3}h^2 - 6Sh + \frac{\sqrt{3}}{4}S^2$. If $\frac{dV}{dh} = 0$, then $h = \frac{6S \pm \sqrt{36S^2 - 4\left(9\sqrt{3}\right)\frac{\sqrt{3}}{4}S^2}}{2\left(9\sqrt{3}\right)} = \frac{6S \pm 3S}{18\sqrt{3}}$. So, we have two solutions to consider, $h = \frac{S}{2\sqrt{3}}$ and $h = \frac{S}{6\sqrt{3}}$. As before, the smaller of the two zeros is the one we want. This can be confirmed by checking the second derivative. If $h = \frac{S}{6\sqrt{3}}$, is the area of the base the same as the area of the sides?

$A_{Base} = \frac{\sqrt{3}}{4}\left(S - 2\sqrt{3}\left(\frac{S}{6\sqrt{3}}\right)\right)^2 = \frac{\sqrt{3}}{4}\left(\frac{2S}{3}\right)^2 = \frac{\sqrt{3}S^2}{9}$ and

$A_{Sides} = 3h\left(S - 2\sqrt{3}h\right) = 3\left(\frac{S}{6\sqrt{3}}\right)\left(S - 2\sqrt{3}\left(\frac{S}{6\sqrt{3}}\right)\right) = \left(\frac{S}{\sqrt{3}}\right)\left(\frac{S}{3}\right) = \frac{S^2}{3\sqrt{3}}$. Once again, the area of the sides is equal to the area of the base when the volume of the "box" is optimized.