Stirling's Formula

When *n* is large, an efficient way to approximate $n!$ is needed. Stirling's formula is often used. Stirling's formula approximates $n!$ with $\sqrt{2\pi n}\, e^{-n}\, n^n$. It is fairly easy to see that a function of the form $e^{-n}\, n^n$ is appropriate. Consider

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

By taking logs we have

$$\ln(n!) = \ln(n) + \ln(n-1) + \ln(n-2) + \cdots + \ln(3) + \ln(2) + \ln(1).$$

The sum on the right side of the equation can be viewed geometrically as the area of the $n-1$ rectangles under the graph of $y = \ln(x)$. This area can be approximated by the area under the curve (see Figure 1).

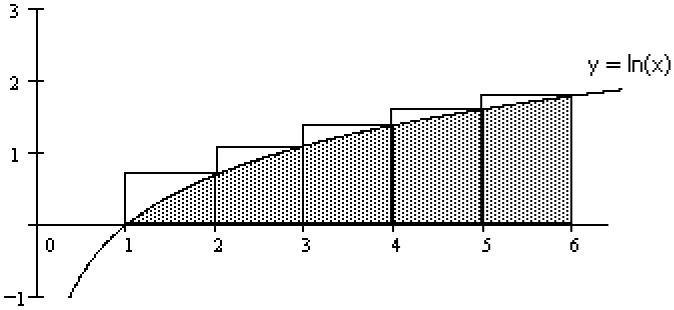

Figure 1

So $\ln(n!) \approx \int_1^n \ln(x)\, dx$. This integral is another standard integration by parts integral. The value is $\int_1^n \ln(x)\, dx = n\ln(n) - n + 1$. For large *n*, the constant 1 is not so important, so $\ln(n!) \approx n\ln(n) - n$. Solving for $n!$, we find

$$(n!) \approx e^{n\ln(n)-n} = e^{-n} e^{\ln(n^n)} = e^{-n} n^n.$$

So something of the form $e^{-n} n^n$ should work. To see where the other terms come from we need a more careful derivation. The following approach is due to mathematician Henry Pollak.

We know that $n! = \int_0^\infty e^{-x} x^n\, dx$ for all integer values of *n*. Rewriting, we have $n! = \int_0^\infty e^{-x} e^{n\ln(x)}\, dx = \int_0^\infty e^{n\ln(x)-x}\, dx$. Consider the function $f(x) = n\ln(x) - x$. It has its maximum value at $x = n$. Approximate the exponent in the integral with its quadratic approximation centered at its maximum value.

$$f(x) = n\ln(x) - x \qquad f(n) = n\ln(n) - n$$

$$f'(x) = \frac{n}{x} - 1 \qquad f'(n) = 0$$

$$f''(x)=\frac{-n}{x^2} \qquad f''(n)=\frac{-1}{n}$$

So, the quadratic approximation of $f(x)=n\ln(x)-x$ at $x=n$ is

$$n\ln(x)-x\approx n\ln(n)-n-\frac{1}{2}\frac{(x-n)^2}{n}.$$

Therefore,

$$n!=\int_0^\infty e^{n\ln(x)-x}\,dx\approx\int_0^\infty e^{n\ln(n)-n-\frac{1}{2}\frac{(x-n)^2}{n}}\,dx=e^{n\ln(n)-n}\int_0^\infty e^{-\frac{1}{2}\frac{(x-n)^2}{n}}\,dx\,.$$

We need to evaluate the definite integral and simplify the constant. The integration is a little tricky, requiring two substitutions. Given $n^n e^{-n}\int_0^\infty e^{\frac{(x-n)^2}{-2n}}\,dx$, we make the substitution $y=x-n$, so we have $n^n e^{-n}\int_{-n}^\infty e^{\frac{y^2}{-2n}}dy$. Now let $u=\frac{y}{\sqrt{n}}$, this gives the integral $n^n e^{-n}\sqrt{n}\int_{-\sqrt{n}}^\infty e^{-\frac{1}{2}u^2}\,du$. For any reasonably large value of *n*, this integral has the same value as $n^n e^{-n}\sqrt{n}\int_{-\infty}^\infty e^{-\frac{1}{2}y^2}\,dy$, since the function $g(x)=e^{-\frac{1}{2}x^2}$ dies out so quickly.

We now have the approximation $n!\approx e^{-n}n^n\sqrt{n}\int_{-\infty}^\infty e^{-\frac{1}{2}u^2}\,du$, which is $n!\approx Ce^{-n}n^n\sqrt{n}$ for some constant *C*. This is the form of Sterling’s formula. This final integral is a variation of the previous standard integral, and has a value of $\int_{-\infty}^\infty e^{-\frac{1}{2}u^2}\,du=\sqrt{2\pi}$.

This gives us Sterling's formula

$$n!\approx e^{-n}n^n\sqrt{2\pi n}\,.$$

This approximation is good inside the region of the quadratic approximation because that approximation is so good. It is good outside the region of the quadratic approximation because the function values are so small.

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