Moment Generating Functions and Their Properties

The *i th moment* of a random variable Y is defined to be $E(Y^i) = m_i'$. So the expected value or mean of Y, $E(Y)$, is the first moment $E(Y^1)$. The expected value of Y^2, $E(Y^2)$, which can be used to find the variance of Y, is the second moment.

The *moment generating function* $m(t)$ for a random variable Y is defined to be $E(e^{tY})$ where t is in a small neighborhood of zero. So $m(t) = E(e^{tY}) = E\left(1 + tY + \frac{(tY)^2}{2!} + \frac{(tY)^3}{3!} + \ldots\right)$ since the series expansion for $e^{tY} = 1 + tY + \frac{(tY)^2}{2!} + \frac{(tY)^3}{3!} + \ldots$. Also, $m'(t) = E\left(Y + \frac{2tY^2}{2!} + \frac{3t^2Y^3}{3!} + \ldots\right)$. Setting $t = 0$, we have $m'(0) = E(Y + 0 + 0 + \ldots) = E(Y)$. So, the first derivative of the moment generating function evaluated at $t = 0$ is the expected value of Y. That is, $m'(0) = E(Y) = m$. Thus, if you have the moment generating function for a random variable, you can find m by taking the first derivative and evaluating it at zero.

Now look at the second derivative:

$$m''(t) = E\left(Y^2 + \frac{3 \cdot 2tY^3}{3!} + \ldots\right), \text{ and } m''(0) = E(Y^2).$$

Since $V(Y) = E(Y^2) - [E(Y)]^2$, we can use the first and second derivatives of the moment generating function to find the variance of a random variable. Before proceeding we will verify that $V(Y) = E(Y^2) - [E(Y)]^2$:

$$\begin{aligned} V(Y) = E(Y - m)^2 &= E(Y^2 - 2Ym + m^2) \\ &= E(Y^2) - E(2mY) + E(m^2) \\ &= E(Y^2) - 2m[E(Y)] + m^2 \\ &= E(Y^2) - 2m(m) + m^2 \\ &= E(Y^2) - 2m^2 + m^2 \\ &= E(Y^2) - m^2 = E(Y^2) - [E(Y)]^2 \end{aligned}$$

The argument given above essentially repeats the calculation for $Cov\left(Y_i, Y_j\right)$ on page 7. Covariance is more general, since $Cov\left(Y, Y\right) = Var\left(Y\right)$.

Note that if Y is a discrete random variable,

$$
\begin{aligned}
m_Y(t) = E(e^{tY}) &= \sum e^{ty} p(y) = \sum p(y)\left[1 + ty + \frac{(ty)^2}{2!} + \ldots\right] \\
&= \sum p(y) + \sum (ty)p(y) + \sum \frac{t^2}{2!} y^2 p(y) + \ldots \\
&= 1 + t\sum yp(y) + \frac{t^2}{2!}\sum y^2 p(y) + \ldots \quad \text{since } t \text{ is constant in the sum} \\
&= 1 + tE(y) + \frac{t^2}{2!}E(y^2) + \ldots \\
&= 1 + tm_1' + \frac{t^2}{2!}m_2' + \ldots
\end{aligned}
$$

as we would expect since this is a moment generating function.

Example 1

Suppose Y is a Poisson random variable with parameter l. Then $p(y) = \dfrac{e^{-l} l^y}{y!}$ for $y = 0,1,2.\ldots$, where l represents the rate at which something happens. Find the moment generating function and use it to find the mean and variance of Y.

Solution:

First find $m_Y(t) = E(e^{tY}) = \displaystyle\sum_{y=0}^{\infty} \frac{e^{ty} e^{-l} l^y}{y!}$

$$
\begin{aligned}
&= e^{-l}\sum_{y=0}^{\infty}\frac{(e^t\, l)^y}{y!} = e^{-l} e^{e^t l} \quad \text{since } \sum_{y=0}^{\infty}\frac{(e^t\, l)^y}{y!} \text{ is the power series for } e^{e^t l} \\
&= e^{e^t l - l} = e^{l(e^t - 1)}
\end{aligned}
$$

$$\therefore m_Y(t) = e^{l(e^t-1)}$$

Find derivatives of $m_Y(t)$ to use for computing m and s^2.

$$m'(t) = e^{l(e^t-1)} \cdot le^t \qquad m''(t) = e^{l(e^t-1)}\, le^t + le^t e^{l(e^t-1)}\, le^t$$

$$E(Y) = m = m'(0) = e^{l \cdot 0}\, le^0 = l \qquad E(Y^2) = m''(0) = e^0\, le^0 + le^0 e^0\, le^0 = l + l^2$$

$$
\begin{aligned}
V(Y) = s^2 &= E(Y^2) - [E(Y)]^2 \\
&= (l + l^2) - l^2 = l
\end{aligned}
$$

The Poisson distribution has mean and variance of l.

Example 2

Suppose Y has a geometric distribution with parameter p. Show that the moment generating function for Y is $m(t) = \dfrac{pe^t}{1 - qe^t}$ where $q = 1 - p$.

Solution:

$$p(y) = q^{y-1}p \qquad y = 1,\ 2,\ldots$$

$$m_Y(t) = E(e^{tY}) = \sum_{y=1}^{\infty} e^{ty} pq^{y-1}$$

$$= pq^{-1}\sum_{y=1}^{\infty}(e^t q)^y$$ Note: geometric series with common ratio $e^t q < 1$ if $t < -\ln(q)$

$$= pq^{-1}\frac{qe^t}{1-qe^t} = \frac{pe^t}{1-qe^t}$$

Now that we know the moment generating function, it is a simple matter to find the mean and variance of the geometric distribution.

For the mean we have

$$m'(t) = \frac{\left(1-qe^t\right)pe^t - pe^t\left(-qe^t\right)}{\left(1-qe^t\right)^2} = \frac{pe^t}{\left(1-qe^t\right)^2}, \text{ so } m'(0) = \frac{pe^0}{\left(1-qe^0\right)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

For the variance we have

$$m''(t) = \frac{\left(1-qe^t\right)^2 pe^t - pe^t(2)\left(1-qe^t\right)\left(-qe^t\right)}{\left(1-qe^t\right)^4} = \frac{pe^t\left(1+qe^t\right)}{\left(1-qe^t\right)^3}$$

and

$$m''(0) = \frac{pe^0\left(1+qe^0\right)}{\left(1-qe^0\right)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2},$$

so $Var(Y) = m''(0) - \left(m'(0)\right)^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$.

Example 3

Find the moment generating function for a random variable with a standard normal distribution. That is, find the moment generating function for $Z \sim N(0,1)$. Note that $Z \sim N(0,1)$ is read: Z is distributed as a normal random variable with $m = 0$ and $s^2 = 1$.

Solution:

$$m_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2p}}e^{\frac{-z^2}{2}}dz$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2p}}e^{\left(\frac{-z^2}{2}+\frac{2tz}{2}-\frac{t^2}{2}+\frac{t^2}{2}\right)}dz$$

$$= e^{t^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2p}}e^{-(z-t)^2/2}dz$$

$$= e^{t^2/2}\int_{-\infty}^{\infty}\left(normal\ density, m = t, s^2 = 1\right)dz$$

$$= e^{t^2/2}\cdot 1 = e^{t^2/2}$$

It is straight forward to verify that the mean and variance are 0 and 1, respectively.

<u>Example 4</u>

Show that the moment generating function for a random variable $Y \sim N(m, s^2)$ is $m_Y(t) = e^{mt+\frac{t^2s^2}{2}}$. Use the moment generating function to show that $E(Y) = m$.

Solution:

Following the outline of Example 3, we have

$$m_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty}\frac{1}{\sqrt{2p}\,s}\ e^{-\frac{1}{2s^2}(y-m)^2}dy$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2p}\,s}\ e^{-\frac{1}{2s^2}(y-m)^2+\frac{2s^2ty}{2s^2}}dy$$

Consider the exponent:

$$-\frac{1}{2s^2}[(y-m)^2-2s^2ty] = -\frac{1}{2s^2}[((y-m)-s^2t)^2-2s^2mt-s^4t^2]$$

So $m_Y(t) = \displaystyle\int_{-\infty}^{\infty}\frac{1}{\sqrt{2p}\,s}\ e^{\frac{-1}{2s^2}[(y-m)-s^2t]^2+mt+\frac{s^2t^2}{2}}dy$

$$= e^{mt+\frac{s^2t^2}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2p}\,s}\ e^{\frac{-1}{2s^2}[(y-m)-s^2t]^2}dy \quad \text{(the integrand is a normal density function)}$$

$$= e^{mt+\frac{s^2t^2}{2}}\int_{-\infty}^{\infty}(normal\ density,\ \text{mean} = m+s^2t, \text{var} = s^2)\ dy$$

$$= e^{mt+\frac{s^2t^2}{2}}\cdot 1 = e^{mt+\frac{s^2t^2}{2}}$$

Now to find the expected value of Y: $m'(t) = e^{mt+\frac{s^2t^2}{2}}\left(m+s^2t\right)$

$$m'(0) = e^0(m+0) = m.$$

The variance of Y is found by computing: $m''(t) = e^{mt+\frac{s^2t^2}{2}}\left(s^2\right)+e^{mt+\frac{s^2t^2}{2}}\left(m+s^2t\right)^2$

$$m''(0) = e^0\left(s^2\right) + e^0\left(m\right)^2 = s^2 + m^2 .$$

Then $Var\left(y\right) = s^2 + m^2 - m^2 = s^2$.

A more direct derivation can also be given:

$$m_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(m+s\,z)}\right) = E\left(e^{mt} \cdot e^{(s\,t)z}\right) = e^{mt} m_Z\left(s\,t\right) = e^{mt} e^{\frac{1}{2}(s\,t)^2} = e^{mt + \frac{1}{2}s^2 t^2} .$$

<u>Example 5</u>

Find the moment generating function for a random variable with a gamma distribution.

Solution:

$$m_Y(t) = E(e^{tY}) = \int_0^\infty \frac{1}{b^a\,\Gamma(a)} y^{a-1} e^{-y/b} e^{ty} dy$$

$$= \frac{1}{b^a \Gamma(a)} \int_0^\infty y^{a-1} e^{-y(\frac{1}{b}-t)} dy \quad \text{(Recall that } b^a \text{ and } \Gamma\left(a\right) \text{ are constants)}$$

Rewrite the expression $\dfrac{1}{b} - t = \dfrac{1 - b\,t}{b} = \dfrac{1}{\left(\dfrac{b}{1 - b\,t}\right)}$

$$= \frac{1}{b^a \Gamma(a)} \int_0^\infty y^{a-1} e^{\frac{-y}{\left(\frac{b}{1-bt}\right)}} dy$$

Since we have previously shown that $\int_0^\infty y^{a-1} e^{-y/b} dy = b^a \Gamma(a)$,

$$= \frac{1}{b^a \Gamma(a)} \left(\frac{b}{1-bt}\right)^a \Gamma(a)$$

$$= \frac{1}{(1-bt)^a} = \left(\frac{1}{1-bt}\right)^a, \text{ if } t < \frac{1}{b}.$$

Recall that the chi-square distribution is a special case of the gamma distribution with $b = 2$ and $a = n/2$. So it follows from Example 5 that if $Y \sim c^2\,(n)$, then $m_Y(t) = \left(\dfrac{1}{1-2t}\right)^{n/2}$.

From this moment generating function we can find the mean and variance of the chi-square distribution with n degrees of freedom.

We have $m_Y'(t) = -\frac{n}{2}(1-2t)^{-\frac{n}{2}-1}(-2) = \frac{n}{(1-2t)^{\frac{n}{2}+1}}$. So $m'(0) = n$. The mean of a chi-square distribution with n degrees of freedom is n, the degrees of freedom.

Also $m''(t) = \frac{2n\left(\frac{n}{2}+1\right)}{(1-2t)^{\frac{n}{2}+2}}$. So $m''(0) = \frac{2n\left(\frac{n}{2}+1\right)}{(1-2t)^{\frac{n}{2}+2}}\Bigg|_{t=0} = n^2 + 2n$ and

$$Var(Y) = n^2 + 2n - n^2 = 2n \ .$$

The variance is twice the degrees of freedom.

Theorem: Consider $Y_1, Y_2, \ldots Y_n$ independent random variables. Let $W = Y_1 + Y_2 + \ldots + Y_n$. Then $m_W(t) = \prod_{i=1}^{n} m_{Y_i}(t)$.

Proof: $m_W(t) = E\left(e^{t\sum Y_i}\right) = E\left(\prod_{i=1}^{n} e^{tY_i}\right)$ by laws of exponents

$= \prod_{i=1}^{n} E\left(e^{tY_i}\right)$ since Y_i are independent

$= \prod_{i=1}^{n} m_{Y_i}(t)$

<u>Example 6</u>

If Y has a binomial distribution with parameters (n, p), show that the moment generating function for Y is $m(t) = \left(pe^t + q\right)^n$ where $q = 1 - p$. Then use the moment generating function to find $E(Y)$ and $V(Y)$.

Solution:

$Y = \sum_{i=1}^{n} X_i$ where the X's are independent and $X_i = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } q \end{cases}$

$m_{X_i}(t) = E(e^{tX_i}) = e^{t\cdot 1}p + e^{t\cdot 0}q = pe^t + q$

$m_Y(t) = m_{X_1}(t) \cdot m_{X_2}(t) \ \ldots \ m_{X_n}(t)$

$= \prod_{1}^{n} (pe^t + q)$

$= [pe^t + q]^n$

Now we can take derivatives to find $E(Y)$ and $V(Y)$.

$$m'(t) = n(pe^t + q)^{n-1} pe^t$$

$$E(Y) = m'(0) = n(p+q)^{n-1} p = np$$

$$m''(t) = n(n-1)(pe^t + q)^{n-2}(pe^t)^2 + n(pe^t + q)^{n-1} pe^t$$

$$E(Y^2) = m''(0) = n(n-1)(p+q)^{n-2} p^2 + n(p+q)^{n-1} p$$

$$\left(n^2 - n\right) p^2 + np = n^2 p^2 + np\left(1-p\right)$$

$$\therefore V\left(Y\right) = E\left(Y^2\right) - \left[E\left(Y\right)\right]^2 = n^2 p^2 + np\left(1-p\right) - \left(np\right)^2$$

$$= np\left(1-p\right)$$

The Uniqueness Theorem is very important in using moment generating functions to find the probability distribution of a function of random variables.

Uniqueness Theorem: Suppose that random variables X and Y have moment generating functions given by $m_X(t)$ and $m_Y(t)$ respectively. If $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

The proof of the Uniqueness Theorem is quite difficult and is not given here.

Example 7

Find the distribution of the random variable Y for each of the following moment-generating functions:

(a) $m(t) = \left(\dfrac{1}{1-2t}\right)^8$ (b) $m(t) = \left[(1/3)e^t + (2/3)\right]^5$

(c) $m(t) = \dfrac{e^t}{2-e^t}$ (d) $m(t) = e^{2\left(e^t - 1\right)}$

Solution:

(a) Since $m(t)$ can be written in the form $\left(\dfrac{1}{1-2t}\right)^{\frac{u}{2}}$ where $u = 16$, Y must be a chi-square random variable with 16 degrees of freedom.

(b) Since $m(t)$ can be written in the form $\left(pe^t + q\right)^n$ where $p = 1/3$ and $n = 5$, Y must be a binomial random variable with $n = 5$ and $p = 1/3$.

(c) Since $m(t)$ can be rewritten as $\dfrac{\left(\frac{1}{2}e^t\right)}{\left(1 - \frac{1}{2}e^t\right)}$, Y must be a geometric random variable with $p = 1/2$.

(d) Since $m(t)$ can be written in the form $e^{l(e^t-1)}$, Y must be a Poisson random variable with $l = 2$.

Theorem: If $W = aY + b$, then $m_W(t) = e^{tb} m_Y(at)$.

Proof:
$$\begin{aligned} m_W(t) &= E(e^{tW}) = E\left(e^{t(aY+b)}\right) \\ &= e^{bt} E(e^{atY}) = e^{bt} m_Y(at) \end{aligned}$$

Theorem: If $W = aY + b$, then $E(W) = aE(Y) + b$ and $V(W) = a^2 V(Y)$

Proof:
$$\begin{aligned} m_W(t) &= e^{bt} m_Y(at) \\ m'_W(t) &= e^{bt} m'_Y(at)\cdot a + m_Y(at) e^{bt}\cdot b \\ m'_W(0) &= m'_Y(0)\cdot a + m_Y(0)\cdot b = aE(Y) + E(e^{0\cdot Y})\cdot b \\ &= aE(Y) + E(1)\cdot b = aE(Y) + b \end{aligned}$$

$$\begin{aligned} m''_W(t) &= ae^{bt} m''_Y(at)\cdot a + m'_Y(at)\cdot ae^{bt}\cdot b + m_Y(at) e^{bt} b^2 + be^{bt} m'_Y(at)\cdot a \\ m''_W(0) &= a^2 m''_Y(0) + abm'_Y(0) + b^2 m_Y(0) + abm'_Y(0) \\ &= a^2 E(Y^2) + 2abE(Y) + b^2 \end{aligned}$$

So
$$\begin{aligned} V(W) &= E(W^2) - \left[E(W)\right]^2 = a^2 E(Y^2) + 2abE(Y) + b^2 - \left[aE(Y) + b\right]^2 \\ &= a^2 E(Y^2) + 2abE(Y) + b^2 - \left[a^2 (E(Y))^2 + 2abE(Y) + b^2\right] \\ &= a^2 E(Y^2) - a^2\left[E(Y)\right]^2 \\ &= a^2\left\{E(Y^2) - \left[E(Y)\right]^2\right\} = a^2 V(Y) \end{aligned}$$

Of course, this theorem can be proven without resorting to moment generating functions, but we present the proof to show that mgf's give the familiar results.

Theorem: If $Z \sim N(0,1)$ and $Y = sZ + m$, then $Y \sim N(m, s^2)$.

Proof: We know that $m_Z(t) = e^{t^2/2}$.

$$\begin{aligned} m_Y(t) &= e^{tu} m_Z(st) \qquad \text{(Applying the theorem proved above)} \\ &= e^{tu} e^{s^2 t^2/2} \\ &= e^{tu + \frac{s^2 t^2}{2}} = \text{ moment generating function of } N(m, s^2) \end{aligned}$$

$\therefore Y \sim N(m, s^2)$ by the uniqueness theorem.

Theorem: If $Y_1 \sim$ Poisson(l_1) and $Y_2 \sim$ Poisson(l_2) and Y_1 and Y_2 are independent, then $W = Y_1 + Y_2$ is Poisson($l_1 +\ l_2$) .

Proof: $m_W(t) = m_{Y_1}(t) \cdot m_{Y_2}(t)$ since Y_1 and Y_2 are independent

$$= e^{l_1\left(e^t-1\right)} e^{l_2\left(e^t-1\right)}$$

$$= e^{\left(l_1+l_2\right)\left(e^t-1\right)}$$

= moment generating function of Poisson($l_1 + l_2$)

$\therefore W \sim Poisson(l_1 + l_2)$ by the uniqueness theorem.

Theorem: Let $W_1, W_2, \ldots, W_n$ be independent chi-square random variables, each with one degree of freedom. Let $W = \sum_{1}^{n} W_i$. Then $W \sim c_n^{\ 2}$.

Proof:

$$m_W(t) = m_{\sum W_i}(t) = \prod_{i=1}^{n} m_{W_i}(t) = \prod_{i=1}^{n} (1-2t)^{-1/2} = (1-2t)^{-n/2}$$

= moment generating function of c^2 with *n* degrees of freedom

$\therefore W \sim c_n^{\ 2}$ by the uniqueness theorem.

We have already considered the mean and variance of a chi-square distribution with n degrees of freedom as a special case of the gamma distribution (see page 14). Now we will give a more direct proof using the chi-square generating function.

Theorem: If $W \sim c_n^{\ 2}$, then $E(W) = n$ and $V(W) = 2n$.

Proof: $m_W(t) = (1-2t)^{\frac{-n}{2}}$

$$m_W'(t) = \frac{-n}{2}(1-2t)^{\frac{-n}{2}-1} \cdot (-2)$$

$$m'(0) = \frac{-n}{2}(1-0)^{\frac{-n}{2}-1} \cdot (-2) = n$$

So $E(W) = n$.

$$m_W''(t) = \left(\frac{-n}{2}\right)\left(\frac{-n}{2}-1\right)(1-2t)^{\frac{-n}{2}-2}(-2)(-2)$$

$$m_W''(0) = 4\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) = 4\left(\frac{n^2}{4}+\frac{n}{2}\right) = n^2 + 2n$$

So $V(W) = E(W^2) - [E(W)]^2 = n^2 + 2n - n^2 = 2n$

Now consider $Z \sim N(0,1)$ and $W = Z^2$. We know the distribution of Z, and now we want to show that the distribution of W is chi-square with one degree of freedom. There are at least two ways to do this. The method of density functions is the harder way, and the method of moment generating functions is the easier way. We will show both in the sections that follow.

Method of Density Functions: Let $F_W(w)$ represent the cumulative distribution function of W for $w > 0$.

$$F_W(w) = P(z^2 \le w) = P(-\sqrt{w} \le z \le \sqrt{w})$$
$$= \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{\sqrt{2p}} e^{-z^2/2} dz = \frac{2}{\sqrt{2p}} \int_0^{\sqrt{w}} e^{-z^2/2} dz$$
$$f_W(w) = \frac{d}{dw} F_W(w) = \frac{2}{\sqrt{2p}} \frac{d}{dw} \int_0^{\sqrt{w}} e^{-z^2/2} dz$$

Let $u = \sqrt{w}$. Then $\dfrac{du}{dw} = \dfrac{1}{2\sqrt{w}}$ and $F(u) = \dfrac{2}{\sqrt{2p}} \displaystyle\int_0^u e^{-z^2/2} dz$.

$$\frac{dF}{du} = \frac{2}{\sqrt{2p}} \frac{d}{du} \int_0^u e^{-z^2/2} dz = \frac{2}{\sqrt{2p}} e^{-u^2/2}$$

By the chain rule, $\dfrac{dF}{dw} = \dfrac{dF}{du} \cdot \dfrac{du}{dw} = \dfrac{2}{\sqrt{2p}} e^{-u^2/2} \dfrac{1}{2\sqrt{w}}$

$$= \frac{2}{\sqrt{2p}} e^{-\left(\sqrt{w}\right)^2/2} \frac{1}{2} w^{-1/2}$$
$$= \frac{1}{\sqrt{2p}} e^{-w/2} w^{-1/2} \quad \text{Note that } \sqrt{p} = \Gamma\left(\frac{1}{2}\right)$$
$$= \frac{1}{2^{1/2}\Gamma\left(\frac{1}{2}\right)} w^{1/2-1} e^{-w/2}, \text{ for } w > 0$$

Note that $f_W(w)$ is the density function for a Gamma random variable with $a = \dfrac{1}{2}$ and $b = 2$. Therefore W is a chi-square random variable with one degree of freedom.

Method of Moment Generating Functions:

$$m_W(t) = E\left(e^{tW}\right) = E\left(e^{tZ^2}\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}} e^{tz^2} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}} e^{\frac{-z^2}{2}(1-2t)} dz$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}} e^{\frac{-z^2}{2(1-2t)^{-1}}} dz$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}} \left[\frac{1}{(1-2t)^{-1/2}} \cdot \frac{1}{(1-2t)^{1/2}}\right] e^{\frac{-z^2}{2(1-2t)^{-1}}} dz$$
$$= \frac{1}{(1-2t)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}(1-2t)^{-1/2}} e^{\frac{-z^2}{2(1-2t)^{-1}}} dz$$

$$= \frac{1}{(1-2t)^{1/2}} \int_{-\infty}^{\infty} \left[\text{Normal pdf with } m = 0 \text{ and } s^2 = (1-2t)^{-1} \right] dz$$

$$= \frac{1}{(1-2t)^{1/2}}, \text{ for } t > 0.$$

By the uniqueness theorem, this is the moment generating function of a chi-square random variable with one degree of freedom.

Now we know that if $Z \sim N(0,1)$, then $Z^2 \sim c_1^{\,2}$. We have previously shown that the sum of n independent chi-square random variables with one degree of freedom is a chi-square random variable with n degrees of freedom. These results lead to the following:
Given $Y_1, Y_2, \ldots Y_n$, independent random variables with $Y_i \sim N(m_i, s_i^{\,2})$ and $Z_i = \dfrac{Y_i - m_i}{s_i} \sim N(0,1)$, then $\sum Z_i^2 \sim c_n^{\,2}$.

The beauty of moment generating functions is that they give many results with relative ease. Proofs using moment generating functions are often much easier than showing the same results using density functions (or in some other way).

Multivariate Moment Generating Functions

Following are several results concerning multivariate moment generating functions:

$$m_{U,V}(s,t) = E(e^{sU+tV})$$

$$m_{U,V}(0,t) = E(e^{tV}) = m_V(t)$$

$$m_{U,V}(s,0) = E(e^{sU}) = m_U(s)$$

The following theorem will be important for proving results in sections that follow:

Theorem: U and V are independent if and only if $m_{U,V}(s,t) = m_U(s) \cdot m_V(t)$.