

Stirling's Formula: $n! \approx \sqrt{2\pi n} e^{-n} n^n$

We know that $n! = \int_0^\infty e^{-x} x^n dx$ for all integer values of n . Rewriting, we have

$$n! = \int_0^\infty e^{-x} e^{n \ln(x)} dx = \int_0^\infty e^{n \ln(x) - x} dx.$$

Consider the function $f(x) = n \ln(x) - x$, which is the exponent in the integrand above. It has its maximum value at $x = n$. Approximate the exponent in the integral with its quadratic approximation centered at its maximum value.

$$f(x) = n \ln(x) - x \quad f(n) = n \ln(n) - n$$

$$f'(x) = \frac{n}{x} - 1 \quad f'(n) = 0$$

$$f''(x) = \frac{-n}{x^2} \quad f''(n) = \frac{-1}{n}$$

So $n \ln(x) - x \approx n \ln(n) - n - \frac{1}{2} \frac{(x-n)^2}{n}$. Therefore,

$$n! = \int_0^\infty e^{n \ln(x) - x} dx \approx \int_0^\infty e^{n \ln(n) - n - \frac{1}{2} \frac{(x-n)^2}{n}} dx = e^{n \ln(n) - n} \int_0^\infty e^{-\frac{1}{2} \frac{(x-n)^2}{n}} dx.$$

We need to evaluate the definite integral and simplify the constant. The integration is a little tricky, requiring two substitutions.

Given $n^n e^{-n} \int_0^\infty e^{-\frac{(x-n)^2}{2n}} dx$, we make the substitution $y = x - n$, so we have

$n^n e^{-n} \int_{-n}^\infty e^{-\frac{y^2}{2n}} dy$. Now let $u = \frac{y}{\sqrt{n}}$, this gives the integral $n^n e^{-n} \sqrt{n} \int_{-\sqrt{n}}^\infty e^{-\frac{1}{2}y^2} dy$. For any

reasonably large value of n , this integral has the same value as $n^n e^{-n} \sqrt{n} \int_{-\infty}^\infty e^{-\frac{1}{2}y^2} dy$, since the function $g(x) = e^{-\frac{1}{2}x^2}$ dies out so quickly.

We now have the approximation $n! \approx e^{-n} n^n \sqrt{n} \int_{-\infty}^\infty e^{-\frac{1}{2}u^2} du$. This latter integral is a standard one, whose value we know to be $\int_{-\infty}^\infty e^{-\frac{1}{2}u^2} du = \sqrt{2\pi}$. This gives us Sterling's formula

$$n! \approx e^{-n} n^n \sqrt{2\pi n}.$$

This approximation is good inside the region of the quadratic approximation because that approximation is so good. It is good outside the region of the quadratic approximation because the function values are so small.