

Discovering Trigonometry

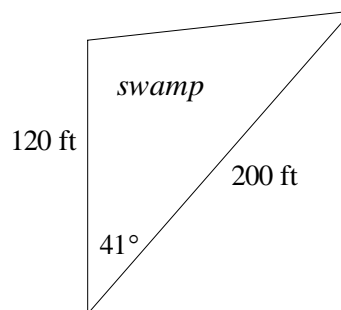
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- 5/3. A 16.0-inch chord is drawn in a circle whose radius is 10.0 inches. What is the *angular size* of the minor arc of this chord? What is the length of the arc, to the nearest tenth of an inch?
- 5/5. What is the area enclosed by a circular *sector* whose radius is r and whose arc length is s ?
- 5/8. A 36-degree counterclockwise rotation centered at the origin sends the point $A = (6,3)$ to the image point A' . To three decimal places, find coordinates for A' .
- 5/10. A sector of a circle is enclosed by two 12.0-inch radii and a 9.0-inch arc. Its perimeter is therefore 33.0 inches. What is the area of this sector, to the nearest tenth of a square inch? What is the central angle of the sector, to the nearest tenth of a degree?
- 5/11. (Continuation) There is another circular sector — part of a circle of a different size — that has the same 33-inch perimeter and that encloses the same area. Find its central angle, radius, and arc length, rounding the lengths to the nearest tenth of an inch.
- 6/14. Write without parentheses: (a) $(xy)^2$ (b) $(x + y)^2$ (c) $(a \sin B)^2$ (d) $(a + \sin B)^2$

- 7/3. A triangular plot of land has the SAS description indicated in the figure at right. Although a swamp in the middle of the plot makes it awkward to *measure* any of the altitudes of this triangle, at least one of them can be *calculated*. Show how. Then use your answer to find the area of the triangle, to the nearest square foot.



- 7/4. (Continuation) Find the length of the third side of the triangle, to the nearest foot.

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7/5. A 15-degree counterclockwise rotation about (4,6) transforms (2,1) onto another point (x,y). Find x and y, correct to three decimal places.

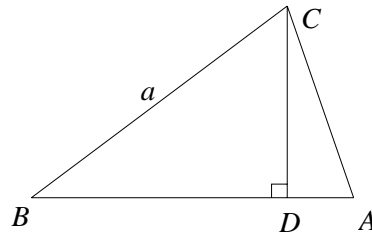
8/1. Simplify: (a) $x \cos^2 \theta + x \sin^2 \theta$ (b) $x \cos^2 \theta + x \cos^2 \theta + 2x \sin^2 \theta$

8/2. A 12.0-cm segment makes a 72.0-degree angle with a 16.0-cm segment. To the nearest tenth of a cm, find the third side of the triangle determined by this SAS information.

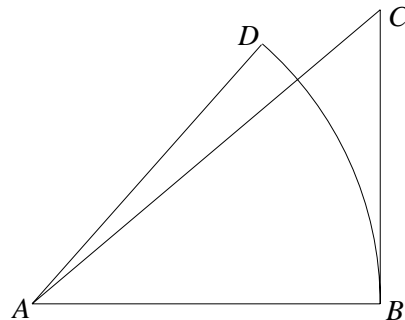
8/3. (Continuation) Find the area of the triangle, to the nearest square centimeter.

8/4. In the diagram at right, CD is the altitude from C .

- (a) Express CD in terms of angle B and side a .
- (b) Express BD in terms of angle B and side a .
- (c) Simplify the expression $(a \sin B)^2 + (a \cos B)^2$ and discuss its relevance to the diagram.
- (d) Why was $a \sin B$ used instead of $\sin B \cdot a$?



8/11. In the figure at right, arc BD is centered at A , and it has the same length as tangent segment BC . Explain why sector ABD has the same area as triangle ABC .



9/12. A circular sector has a 8.26-inch radius and a 12.84-inch arc length. There is another sector that has the same area and the same perimeter. What are its measurements?

9/13. (Continuation) Given a circular sector, is there always a different sector that has the same area and the same perimeter? Explain your answer.

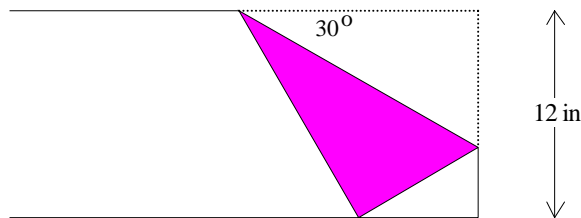
10/1. Solve for y : $x^2 = a^2 + b^2 - 2aby$

10/2. A segment that is a units long makes a C -degree angle with a segment that is b units long. In terms of a , b , and C , find the third side of the triangle defined by this SAS description. You have done numerical versions of this question. Start by finding the length of the altitude drawn to side b , as well as the length of the perpendicular projection of side a onto side b . The resulting formula is known as the *Law of Cosines*.

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10/3. (Continuation) What is the area of the triangle defined by a , b , and C ?

10/4. The figure at right shows a long rectangular strip of paper, one corner of which has been folded over to meet the opposite edge, thereby creating a 30-degree angle. Given that the width of the strip is 12 inches, find the length of the crease.



10/5. (Continuation) Suppose that the size of the folding angle is θ degrees. Use trigonometry to express the length of the crease as a function of θ . Check using the case $\theta = 30$.

10/6. (Continuation) Find approximately that value of θ that makes the crease as short as it can be. Restrict your attention to angles that are smaller than 45 degrees. (Why is this necessary?)

10/7. A triangle has an 8-inch side, a 10-inch side, and an area of 16 square inches. What can you deduce about the angle formed by these two sides?

10/11. Let $\mathbf{u} = [a, b, c]$, $\mathbf{v} = [p, q, r]$, and $\mathbf{w} = [k, m, n]$ for the following questions:

- Verify that $\mathbf{u} \cdot \mathbf{v}$ is the same number as $\mathbf{v} \cdot \mathbf{u}$, for any vectors \mathbf{u} and \mathbf{v} .
- What is the significance of the number $\mathbf{u} \cdot \mathbf{u}$?
- What does the equation $\mathbf{u} \cdot \mathbf{v} = 0$ tell us about the vectors \mathbf{u} and \mathbf{v} ?
- Is it true that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ holds for all vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} ?
- If \mathbf{u} and \mathbf{v} represent the sides of a parallelogram, then $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ represent the diagonals. Justify this, then explain what the equation $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$ tells us about the parallelogram. Give an example of two nonzero vectors \mathbf{u} and \mathbf{v} that fit this equation.

11/2. Find the third side of a triangle in which a 4.00-inch side and a 6.00-inch side are known to make a 56.0-degree angle. Round your answer to three significant digits.

11/4. The radius of the circumscribed circle of the triangle ABC is 15 cm. Given that B is a 49-degree angle, find the length of side AC .

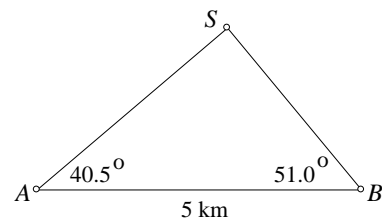
11/5. (Continuation) The radius of the circumscribed circle of the triangle ABC is r cm. Given that B is a θ -degree angle, find the length of side AC , in terms of r and θ .

11/7. The perimeter of a triangle, its area, and the radius of the circle inscribed in the triangle are related in an interesting way. Prove that the radius of the circle times the perimeter of the triangle equals twice the area of the triangle.

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- 11/10. Given a vector \mathbf{u} , the familiar *absolute-value* notation $|\mathbf{u}|$ is often used for its *magnitude*. Thus the expressions $\mathbf{u} \cdot \mathbf{u}$ and $|\mathbf{u}|^2$ both mean the same thing. What exactly *do* they mean?
- 11/11. A familiar algebraic result is that, for any two numbers a and b , the product of $a - b$ times itself is equal to $a^2 - 2ab + b^2$. Does the analogous result hold for dot products of a vector $\mathbf{u} - \mathbf{v}$ with itself? In other words, is it true that $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}$? Justify your conclusion, trying not to express vectors \mathbf{u} and \mathbf{v} in component form.
- 11/12. A triangle has a 56-degree angle, formed by a 10-inch side and an x -inch side. Given that the area of the triangle is 18 square inches, find x .
- 12/4. Triangle ABC has a 63.0-degree angle at B , and side AC is 13.6 cm long. What is the diameter of the circle circumscribed about ABC ?
- 12/5. (Continuation) Given any triangle ABC , with sides a , b , and c opposite angles A , B , and C , respectively, what can be said about the three ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$? This result is known as the *Law of Sines*.

- 12/8. Two observers who are 5 km apart simultaneously sight a small airplane flying between them. One observer measures a 51.0-degree inclination angle, while the other observer measures a 40.5-degree inclination angle. At what altitude is the airplane flying?



- 12/9. If a triangle has sides of lengths a and b , which make a C -degree angle, then the length of the side opposite C is c , where $c^2 = a^2 + b^2 - 2ab \cos C$. This is the SAS version of the Law of Cosines. Explain the terminology. Derive an equivalent SSS version of the Law of Cosines, which gives the cosine of the angle in terms of the lengths of the three sides. Now use it to find the angles of the triangle whose sides have lengths 4 cm, 5 cm, and 6 cm.
- 13/1. Suppose that two vectors \mathbf{u} and \mathbf{v} fit the equation $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}$. How must these vectors \mathbf{u} and \mathbf{v} be related? What familiar theorem does this equation represent?
- 13/5. A triangle has a 5-inch side and an 8-inch side, which form a 60-degree angle. Find
- the area of this triangle;
 - the length of the projection of the 8-inch side onto the 5-inch side;
 - the length of the third side of this triangle;

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- (d) the sizes of the other two angles of this triangle;
- (e) the length of the median drawn to the 8-inch side;
- (f) the length of the bisector of the angle opposite the 8-inch side;
- (g) another triangle that has a 5-inch side, an 8-inch side, and the same area as this triangle.

13/7. Triangle PEA has a 20-degree angle at P and a 120-degree angle at E , and the length of side EA is 6 inches. Find the lengths of the other two sides of this triangle.

13/8. Let $\mathbf{u} = [2, -3, 1]$ and $\mathbf{v} = [0, 1, 4]$. Calculate the vector $\mathbf{u} - \mathbf{v}$. Place \mathbf{u} and \mathbf{v} tail-to-tail to form two sides of a triangle. With regard to this triangle, what does $\mathbf{u} - \mathbf{v}$ represent? Calculate the number $\mathbf{u} \cdot \mathbf{u}$ and discuss its relevance to the picture you have drawn. Finally, calculate the product $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$ and explain its significance.

14/1. The lengths QR , RP , and PQ in triangle PQR are often denoted p , q , and r , respectively. What do the formulas $\frac{1}{2}pq \sin R$ and $\frac{1}{2}qr \sin P$ mean? Justify the equation $\frac{1}{2}pq \sin R = \frac{1}{2}qr \sin P$, then simplify it to a familiar form.

14/5. An isosceles triangle has two sides of length w that make a 2α -degree angle. Write down two different formulas for the *area* of this triangle, in terms of w and α . By equating the formulas, discover a relation involving $\sin 2\alpha$, $\sin \alpha$, and $\cos \alpha$.

14/6. A parallelogram has a 7-inch side and a 9-inch side, and the longer diagonal is 14 inches long. Find the length of the other diagonal. Do you *really* need your calculator for this one?

Bonus: Are there any other quadrilaterals for which the sum of the squares of the diagonals equals the sum of the squares of the sides?

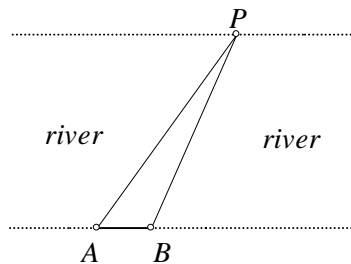
- 14/9. (a) Let $\mathbf{u} = [2, 1]$ and $\mathbf{v} = [1, -3]$. Find the angle formed by \mathbf{u} and \mathbf{v} .
(b) Let $\mathbf{u} = [-1, 0, 1]$ and $\mathbf{v} = [0, 2, -2]$. Find the angle formed by \mathbf{u} and \mathbf{v} .

14/12. There are two noncongruent triangles that have a 9-inch side, a 10-inch side, and that enclose 36 square inches of area. Find the length of the third side in each of these triangles.

- 15/3. Draw vectors \mathbf{u} and \mathbf{v} tail-to-tail so that they make a θ -degree angle. Draw the vector $\mathbf{u} - \mathbf{v}$, the third side of the triangle, and check to see that it points in the right direction.
- (a) Solve for $\cos \theta$ using the SSS version of the Law of Cosines, expressing all lengths in terms of \mathbf{u} , \mathbf{v} , and $\mathbf{u} - \mathbf{v}$.
 - (b) If you use vector algebra to simplify the numerator as much as possible, you will discover an interesting new result connecting $\mathbf{u} \cdot \mathbf{v}$ to $\cos \theta$.

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- 18/2. Two industrious PEA students are trying to find the distance across the Squamscot River. After marking points A and B sixty meters apart on one bank, they sight the Powderhouse P on the opposite bank, and measure angles PAB and PBA to be 54 and 114 degrees, respectively. This enables them to calculate the altitude from P to the baseline AB . To the nearest meter, what was their result?



- 18/7. Two fire wardens are stationed at locations P and Q , which are 45.0 km apart. A forest fire F is sighted by each of the wardens. Given that angle FPQ is 52.0 degrees and angle FQP is 43.0 degrees, find the distance from F to the nearer warden, to the nearest tenth of a km.

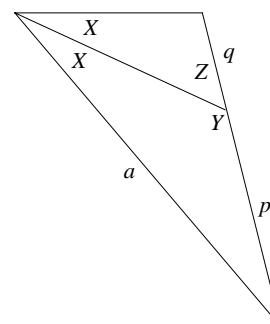
- 18/8. Find the area of a triangle that has a 10 -inch side, a 17 -inch side, and a 21 -inch side.

- 19/3. Apply the 57 -degree counterclockwise rotation about the origin to the vectors $[1,0]$ and $[0,1]$, then use the image vectors (*written as columns*) to form the coefficient matrix \mathbf{M} for the rotation. Test \mathbf{M} by calculating the products $\mathbf{M} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{M} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Where does this rotation send the vector $[3,1]$? Does \mathbf{M} , when applied to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, do its job correctly?

- 19/4. Write the coefficient matrix for a θ -degree counterclockwise rotation about the origin.

- 19/5. Let $A = (0,0,0)$, $B = (9,8,12)$, and $C = (6,2,3)$. Find coordinates for the point on line AB that is closest to C .

- 21/1. Refer to the diagram at right for the following questions: Express the ratio $p:a$ in terms of $\sin X$ and $\sin Y$. Express the ratio $q:c$ in terms of $\sin X$ and $\sin Z$. Because angles Y and Z are supplementary, you can now combine the preceding answers to obtain a familiar result about angle bisectors.



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21/4. The matrix product $\begin{bmatrix} \cos 32 & -\sin 32 \\ \sin 32 & \cos 32 \end{bmatrix} \begin{bmatrix} \cos 40 & -\sin 40 \\ \sin 40 & \cos 40 \end{bmatrix}$ is equivalent to $\begin{bmatrix} \cos 72 & -\sin 72 \\ \sin 72 & \cos 72 \end{bmatrix}$. Verify this statement, then explain why this result could have been expected.

22/1. Calculate the matrix product $\begin{bmatrix} \cos 47 & -\sin 47 \\ \sin 47 & \cos 47 \end{bmatrix} \begin{bmatrix} \cos 47 & \sin 47 \\ -\sin 47 & \cos 47 \end{bmatrix}$. Interpret the result.

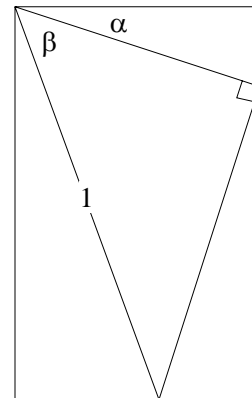
22/2. A parallelogram has a 5-inch side and an 8-inch side that make a 50-degree angle. Find the area of the parallelogram and the lengths of its diagonals.

22/4. Find the diameter of the circle that can be circumscribed around a triangle that has two 13-inch sides and one 10-inch side.

22/5. What does the SAS version of the Law of Cosines have to say about the “triangle” whose sides p and q form a 180-degree angle?

22/9. Triangle KLM has a 120-degree angle at K and side KL is three fourths as long as side LM . To the nearest tenth of a degree, find the sizes of the other two angles of this triangle.

23/2. The rectangle shown has been formed by fitting together four right triangles. As marked, the sizes of two of the angles are α and β (Greek “alpha” and “beta”), and the length of one segment is 1. Find the two unmarked angles whose sizes are α and $\alpha + \beta$. By labeling all the segments of the diagram, discover formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, written in terms of $\sin \alpha$, $\cos \alpha$, $\sin \beta$, and $\cos \beta$.



23/5. Consider the points $A = (\cos 40, \sin 40)$, $B = (\cos 76, \sin 76)$, $C = (\cos 121, \sin 121)$, and $D = (\cos 157, \sin 157)$. Find the lengths of segments AB and CD , then explain what is predictable about the answer.

23/9. A triangle has a 13-inch side, a 14-inch side, and a 15-inch side. To the nearest tenth of an inch, how long is the median drawn to the 14-inch side?

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25/4. What does the SAS version of the Law of Cosines have to say about the “triangle” whose sides p and q form a 0-degree angle?

26/1. What is the radius of the circumscribed circle for a triangle whose sides are 13, 14, and 15?

27/4. Describe the transformation whose matrix is given by $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$.

27/8. A spider is on the rim of an empty conical cup when it spies a fly one third of the way around the rim. The cone is 36 cm in diameter and 24 cm deep. In a hurry for lunch, the spider chooses the shortest path to the fly. How long is this path?

28/2. Simplify the matrix product $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}$.

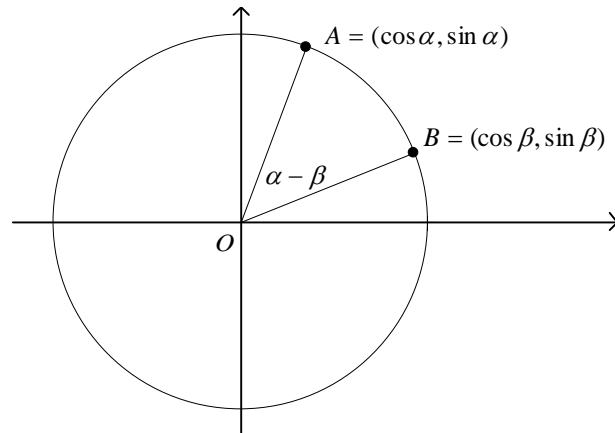
31/4. Let \mathbf{R} be the 36-degree counterclockwise rotation around the origin. Let P' be the point that results when \mathbf{R} is applied to $P = (5,2)$. Find coordinates for P' without using a matrix, then find coordinates for P' by using a matrix. If you had to apply \mathbf{R} to *many* points, which method would you prefer?

31/8. In the diagram, points A and B are on the unit circle, and O is the origin.

(a) Explain why the central angle BOA is labeled $\alpha - \beta$.

(b) Obtain a formula for $\cos(\alpha - \beta)$ by applying the Law of Cosines in its vector form.

(c) Replace β by $-\beta$ in your formula to obtain a familiar formula for $\cos(\alpha + \beta)$.



32/5. There is more than one triangle PQR that fits the description $p = 13$, $r = 14$, $\sin P = 4/5$. For each triangle, find q (the length of the third side), the sizes of the angles, and make a sketch.

34/1. Evaluate $[\cos \theta, \sin \theta] \cdot [\cos(90 + \theta), \sin(90 + \theta)]$. Explain why the value is independent of θ .

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34/5. The matrix $\mathbf{M} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$ defines an isometry of the xy -plane.

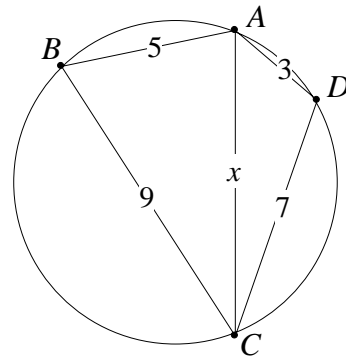
- (a) What special properties do the column vectors of this matrix have?
- (b) Verify that the point $(2,4)$ remains stationary when \mathbf{M} is applied to it.
- (c) What is the significance of the stationary point $(2,4)$? What does it tell us about the possible isometries that \mathbf{M} could represent? Are there other points that invite examination?
- (d) Show that $\mathbf{M}\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What does this suggest about the geometric transformation that \mathbf{M} represents? Confirm your suspicions.

35/4. The sides of a triangle are 10, 17, and 21 inches long. Find

- (a) the smallest angle of the triangle;
- (b) the diameter of the circumscribed circle.

35/9. Quadrilateral $ABCD$ is inscribed in a circle, and the lengths of its sides are $AB = 5$, $BC = 9$, $CD = 7$, and $DA = 3$. Let x be the unknown length of diagonal AC .

- (a) In terms of x , write an expression for $\cos B$.
- (b) In terms of x , write an expression for $\cos D$.
- (c) A simple relationship holds between angles B and D . Use it to help you find the unknown length x .



35/10. The value of $[\cos \theta, \sin \theta] \cdot [\cos(180 + \theta), \sin(180 + \theta)]$ does not depend on the value of θ . Explain why.

37/8. Find the area of the triangle whose vertices are $A = (4,7,1)$, $B = (12,-1,5)$, and $C = (2,-3,12)$.

46/6. The value of $[\cos \theta, \sin \theta] \cdot [\cos(\beta + \theta), \sin(\beta + \theta)]$ does depend on the value of β , but it does not depend on the value of θ . Explain why.