Swinging Ellipses

Adapted from *Real-World Math with the CBL System, Texas Instruments*

In this activity students will use a CBL data collection device with the motion detector to collect harmonic motion data and analyze the data using Precalculus methods. This problem provides a connection between trigonometric functions, inverse trigonometric functions and conic sections.

Part I - Practice writing trigonometric functions from a graph and then consider the velocity functions as a function of position.

Students need to be able to:

- Write trigonometric functions given the graph.
- Solve trigonometric equations for one variable in terms of the other.
- Re-write trigonometric expressions containing trigonometric functions and inverse trigonometric functions so that no trigonometric functions appear in the final expression.
- Write the equation of an ellipse.

Equipment:

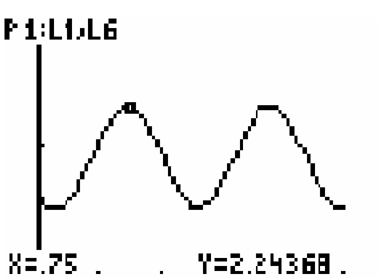
- CBL or CBR and motion detector
- TI-83, 84 or 89 with unit to unit link cable
- DataMate Program – you can download DataMate from the TI website
- Ring stand, a pendulum bob and string

Part I Finding a Model for Position

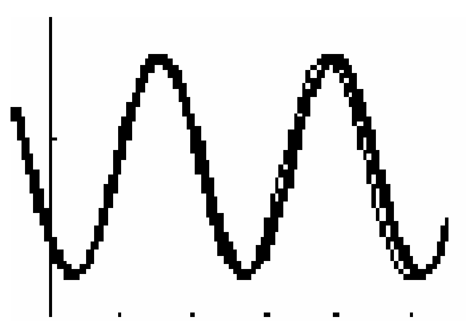
A sample of the distance data versus time is shown below. Note: The program stores time in L1, distance in L6 and velocity in L7.

The students should write a function that describes the horizontal position of the pendulum at time t. Let $x(t)$ be the horizontal position of the pendulum in feet, where t is measured in seconds. The functions will be of the form: $x(t) = a\sin(b(t+c)) + d$ or $x(t) = a\cos(b(t+c)) + d$.

We have $x(t) = -0.315\cos\left(\frac{2\pi}{1.2}(t-0.15)\right) + 1.925$. (1)

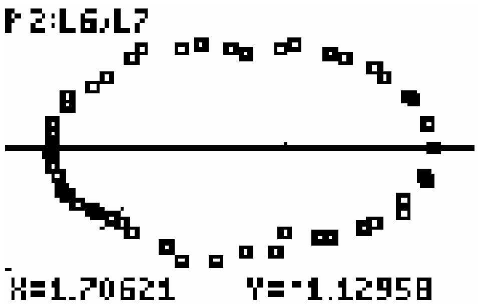


The graph of the function with the data is shown below:



Part II - Using Geometry to Find a Model for Velocity Versus Position

Next we want to consider the graph of the velocity of the pendulum. We'd like to plot velocity against the position of the pendulum, so we will turn on a STAT PLOT of L6 against L7. That data is shown below:
This relationship is elliptical. We can find the equation of this ellipse using our knowledge of conic sections.

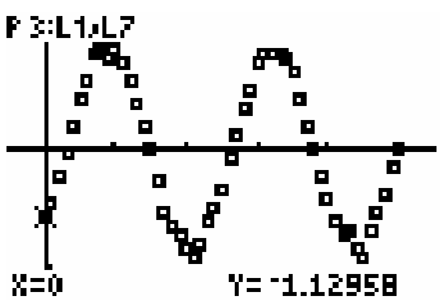


The equation is given as $\frac{(x-c)^2}{a^2}+\frac{y^2}{b^2}=1$ where (c,0) is the center of the ellipse and $2a$ and $2b$ are the lengths of the minor and major axes respectively.

Part III - Using Trigonometry to Find a Model for Velocity Versus Position

Another way to find the equation for the ellipse is to
1. Find a model for the velocity as a function of time.
2. Using our model for the position of the pendulum, solve for *t* in terms of *x*.
3. Substitute for *t* in the velocity equation and simplify.

Given the velocity of the pendulum below:



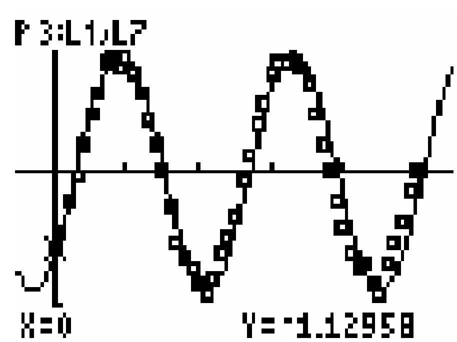
Through methods similar to Part I of the activity, we can find $v(t)$.

Note: Since we used the **reflected cosine** function for position, we will use the sine function to model the velocity. This is because the velocity is the derivative of the position, and the derivative of $-\cos(t)$ is $\sin(t)$.

The velocity function is

$$v(t)=1.79913\sin\left(\frac{2\pi}{1.2}(t-0.15)\right). \qquad (2)$$

To the right is a graph of the velocity function with the data:



Now, we want to solve equation (1) in Part I for *t* in terms of *x*.

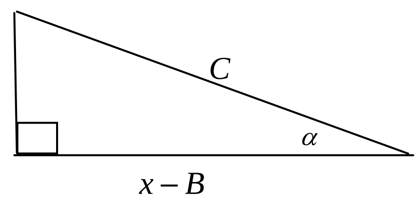
$$x(t) = 0.315\cos\left(\frac{2\pi}{1.2}(t-0.15)\right) + 1.925 \qquad \text{so} \quad t = \frac{1.2}{2\pi}\cos^{-1}\left(\frac{x-1.925}{0.315}\right) + 0.15$$

This expression looks very messy, but when we substitute the expression for *t* into equation (2), we get:

$v(x) = 1.7993\sin\left(\cos^{-1}\left(\frac{x-1.925}{0.315}\right)\right)$. We have accomplished our goal of writing *v* as a function of *x*. According to our data, the graph of *v* versus *x* should be that of an ellipse, but if you graph this function, you will only see the "top half" of the ellipse and we would like to re-write this equation so that we can recognize that the graph is an ellipse.

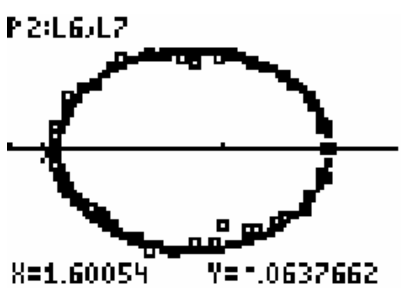
We will re-write our expression to make it easier to manipulate, so let

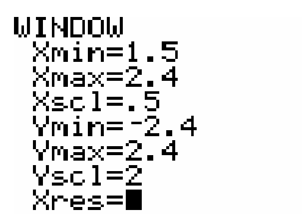
$v(x) = A\sin\left(\cos^{-1}\left(\frac{x-B}{C}\right)\right)$. We need to re-write the expression $\sin\left(\cos^{-1}\left(\frac{x-B}{C}\right)\right)$ without trigonometric functions. Let $\alpha = \cos^{-1}\left(\frac{x-B}{C}\right)$ and $\cos(\alpha) = \frac{x-B}{C}$. We can draw a right triangle and label its sides and the angle α as shown and



So $\sin(\alpha) = \frac{\sqrt{C^2-(x-B)^2}}{C}$, and $v = A\frac{\sqrt{C^2-(x-B)^2}}{C}$ or $\frac{v^2}{A^2} + \frac{(x-B)^2}{C^2} = 1$

Now we can recognize the equation of the ellipse. $\frac{v^2}{0.567^2} + \frac{(x-1.925)^2}{0.315^2} = 1$





What a beautiful fit!

If you have questions or would like the complete solutions and a student handout, please send e-mail to hernandez@ncssm.edu or robinson@ncssm.edu