Maximizing Triangle Area
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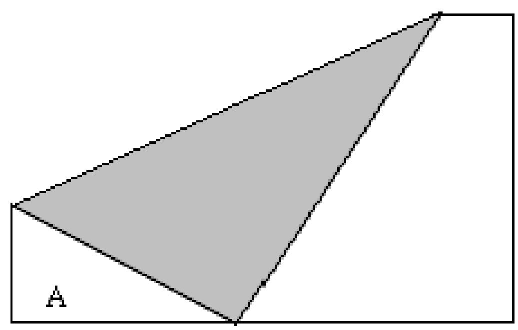
Following is the problem we worked on early this year in Precalculus:

Take a sheet of paper and fold the upper left corner so that it touches some point on the bottom edge, as illustrated here. Consider the area of the triangle labeled "A" – the one formed in the lower left corner of the paper. Find the dimensions of the triangle that has the largest area.



Students worked in groups to collect data; each group had different-sized sheet of paper. Expectations included a written report that included both a "data solution" and an "algebraic solution." This talk looks at those solutions, plus extensions involving more data analysis and calculus.

The full handout is available online:

http://math.ncssm.edu/~rash/tcm2008

or

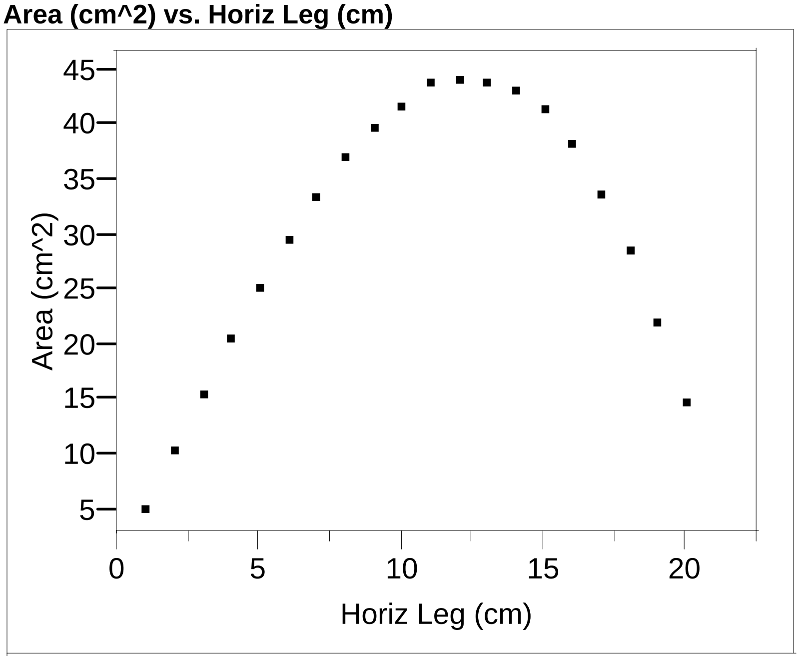
http://courses.ncssm.edu/math/TCMConf/TCM%202008/Talks/talks2008index.htm

Let's begin with a sample data set collected from folding an 8.5" x 11" sheet of paper:

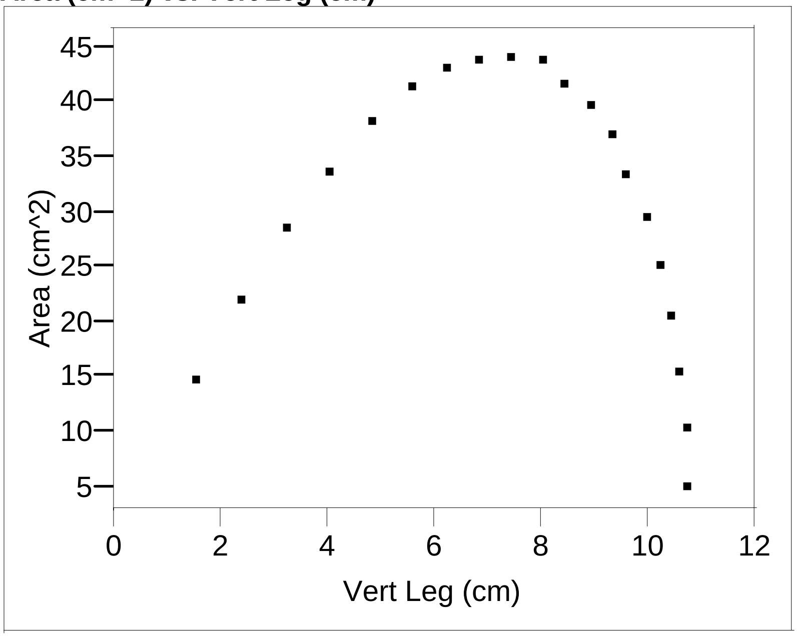
Horiz. leg (cm)	Vert. leg (cm)	Area (cm^2)
1	10.7	5.35
2	10.7	10.7
3	10.55	15.825
4	10.4	20.8
5	10.2	25.5
6	9.95	29.85
7	9.6	33.6
8	9.3	37.2
9	8.9	40.05
10	8.4	42
11	8	44
12	7.4	44.4
13	6.8	44.2
14	6.2	43.4
15	5.55	41.625
16	4.8	38.4
17	4	34
18	3.2	28.8
19	2.35	22.325
20	1.5	15

And some plots of our data:

Area (cm^2) vs. Horiz Leg (cm)



Area (cm^2) vs. Vert Leg (cm)

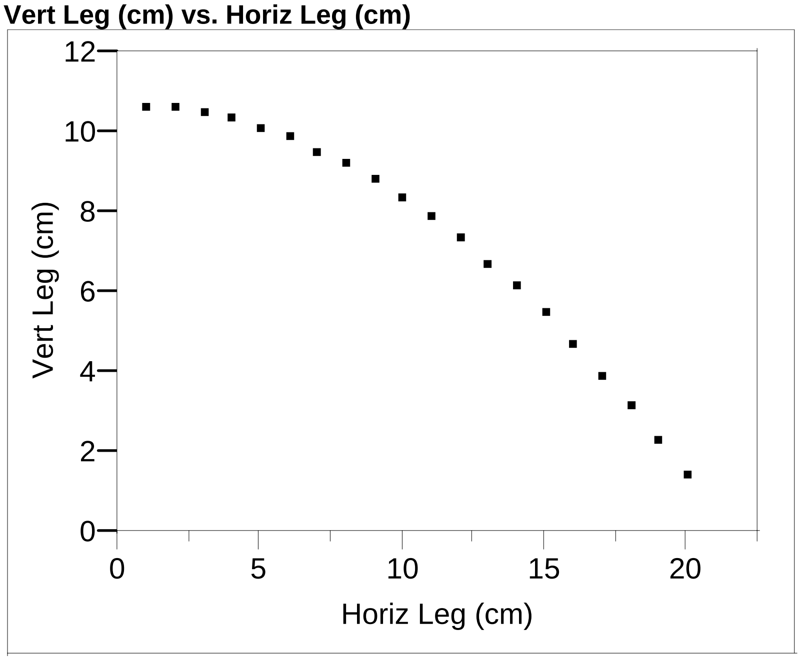


By inspecting our data and scatterplot, we see the maximum area is approximately 44.4 cm^2, when the dimensions are approximately 12 cm x 7.4 cm.

What functions best describe the relationships in the scatterplots above? Quadratic? Some other polynomial? Something else? (Will revisit this question soon…)

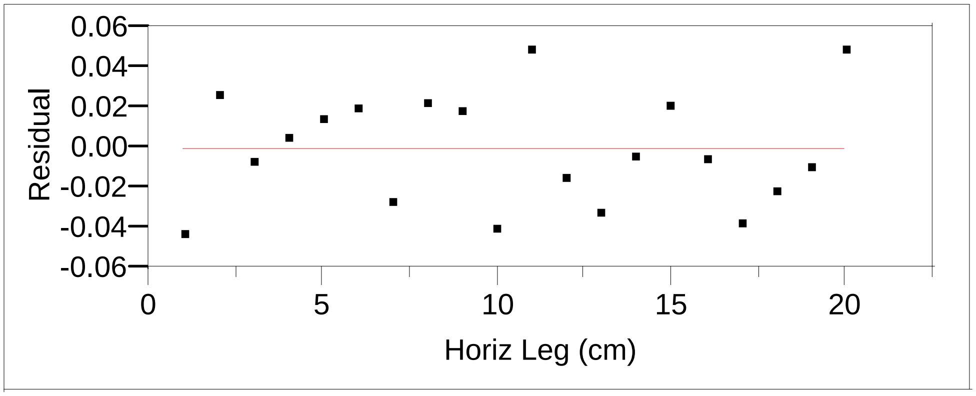
Or, can we predict the vertical leg length if we know the horizontal leg length?

Vert Leg (cm) vs. Horiz Leg (cm)



Seems to be parabolic? (The quadratic fit on the next page seems promising – the residuals show no obvious pattern!)

Bivariate Fit of Vert Leg (cm) By Horiz Leg (cm)
Polynomial Fit Degree=2
Vert Leg (cm) = 10.7646 - 0.000571 Horiz Leg (cm) - 0.023314 (Horiz Leg (cm))^2



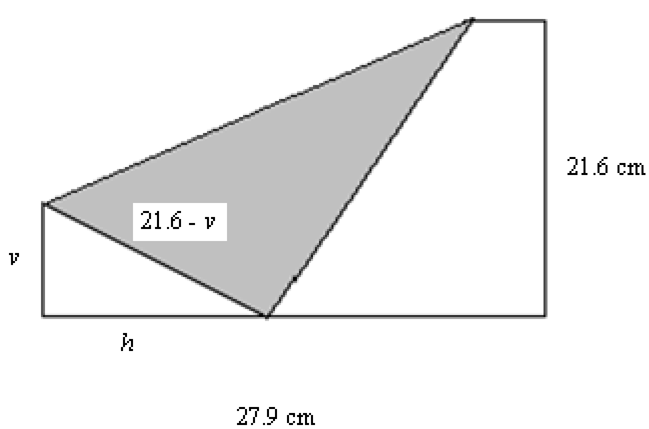
So, we could write $A(h,v) = \frac{1}{2}hv$, and with $v(h) = 10.7646 - .000571h - 0.023314h^2$, we have $A(h) = \frac{1}{2}h * (10.7646 - .000571h - 0.023314h^2)$. (Where v and h are the vertical and horizontal legs of the triangle, respectively.) Then if we graph this function, we notice it matches our "area vs horiz. leg" scatterplot well and we can use our calculator to find the maximum area of 44.56 cm^2, with a horiz. leg length of 12.4 cm. Then $v(12.4) = 7.18$ cm would be the vertical leg length.

What about domain?

Notice that our model $A(h)$ (as given above) is a cubic polynomial. Polynomials have domains of all real numbers, but is this a sensible domain in the context of our problem? It would only make sense to have $A(h) > 0$, and a quick inspection reveals that for this to be the case, we must have $0 < h < 21.5$ (approximately). (Why 21.5? Notice this is approximately the shorter dimension of the sheet of paper.)

Algebraic Approach

Realizing that the hypotenuse of our right triangle is simply the difference between the shorter side of the sheet of paper and the triangle's vertical leg, we can label our figure as such:



Then starting with Pythagorean Theorem, we have

$$h^2 + v^2 = (21.6 - v)^2$$
$$h^2 + v^2 = 21.6^2 - 2(21.6)v + v^2$$
$$h^2 - 21.6^2 = -2(21.6)v$$
$$v(h) = \frac{21.6^2 - h^2}{2(21.6)}$$

Then, since $A(h, v) = \frac{1}{2}hv$, we have

$$A(h) = \frac{1}{2}h\left(\frac{21.6^2 - h^2}{2(21.6)}\right)$$

Notice that in order to have $A(h) > 0$ we need $0 < h < 21.6$.

Graphing this function in our calculator, we find the maximum area of 44.9 cm^2, when the horizontal leg is 12.47 cm.

What if we had chosen to write our model for area in terms of the vertical leg instead?

Then we would write $h(v) = \sqrt{(21.6 - v)^2 - v^2} = \sqrt{21.6^2 - 2(21.6)v}$, and

$$A(v) = \frac{1}{2}v\sqrt{21.6^2 - 2(21.6)v}$$

Of course we still find the same maximum area of 44.9 cm^2, but with a vertical leg length of 7.2 cm. Notice that while $A(h)$ is simply a cubic function, $A(v)$ is not even a polynomial; perhaps something different than what Precalculus students may be used to, and thus explaining the slightly different shapes of our earlier scatterplots of $A(h)$ and $A(v)$. (Notice for $A(v)$, the "domain of reality is $0 < v < 21.6/2$.

An Extension

What if we used a different-sized sheet of paper?

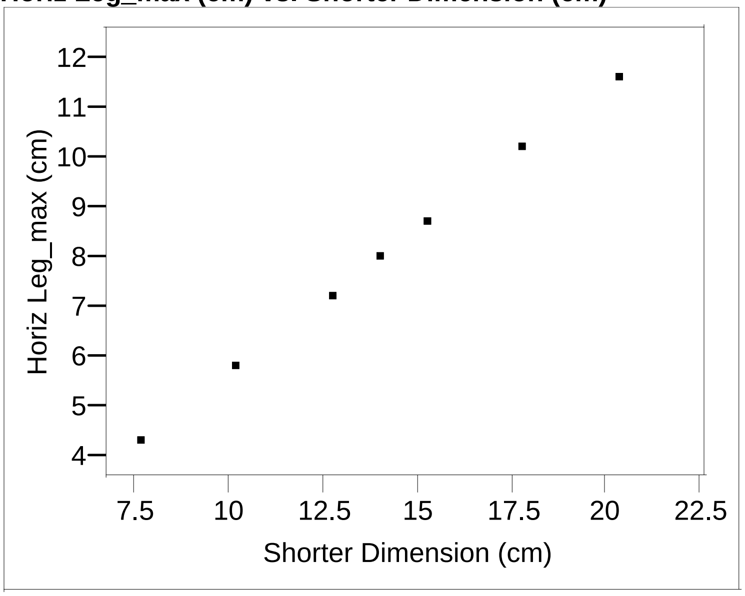
Redo the activity with various sizes of paper and collect data on the dimensions that maximize the triangle's area. Some sample data follows:

Shorter side (cm)	Longer side (cm)	Horiz. leg $_{max}$ (cm)	Vert. leg $_{max}$ (cm)	Area (cm^2)
7.62	12.7	2.5	4.4	5.5
10.16	12.7	3.4	5.9	10.03
7.62	15.24	2.5	4.4	5.5
12.7	15.24	4.2	7.3	15.33
10.16	17.78	3.4	5.9	10.03
12.7	17.78	4.2	7.3	15.33
13.97	17.78	4.7	8.1	19.035
15.24	20.32	5.1	8.8	22.44
13.97	21.59	4.7	8.1	19.035
15.24	22.86	5.1	8.8	22.44
17.78	25.4	5.9	10.3	30.385

17.78	27.94	5.9	10.3	30.385
20.32	27.94	6.8	11.7	39.78
20.32	30.48	6.8	11.7	39.78

Can we predict the horizontal leg length that maximizes the area from one of the side lengths of the paper?

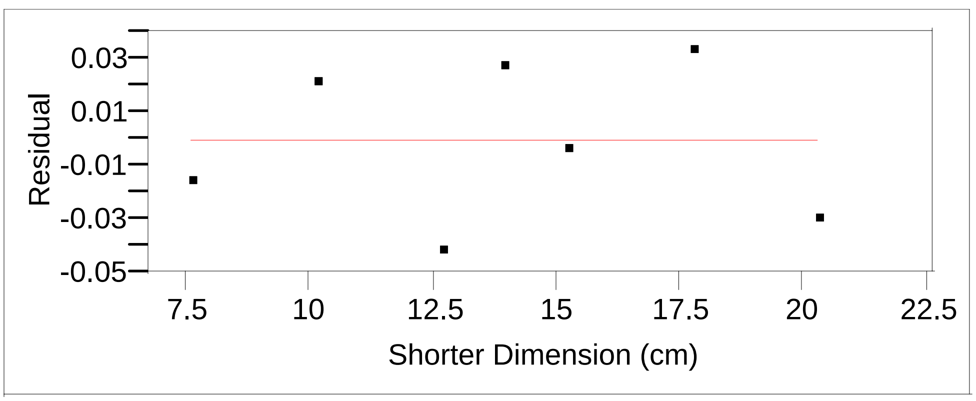
Horiz Leg_max (cm) vs. Shorter Dimension (cm)



Seems linear; let's try a linear fit and check the residuals:

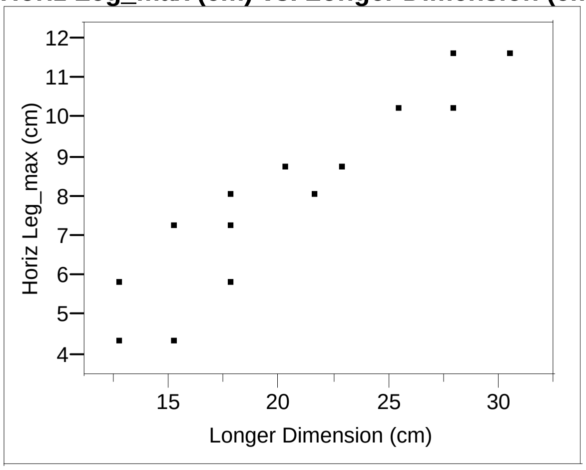
Linear Fit

Horiz Leg_max (cm) = 0.0257143 + 0.575928 Shorter Dimension (cm)



Looks good; what if we had tried to predict the horizontal leg length that maximizes the area from the longer dimension?

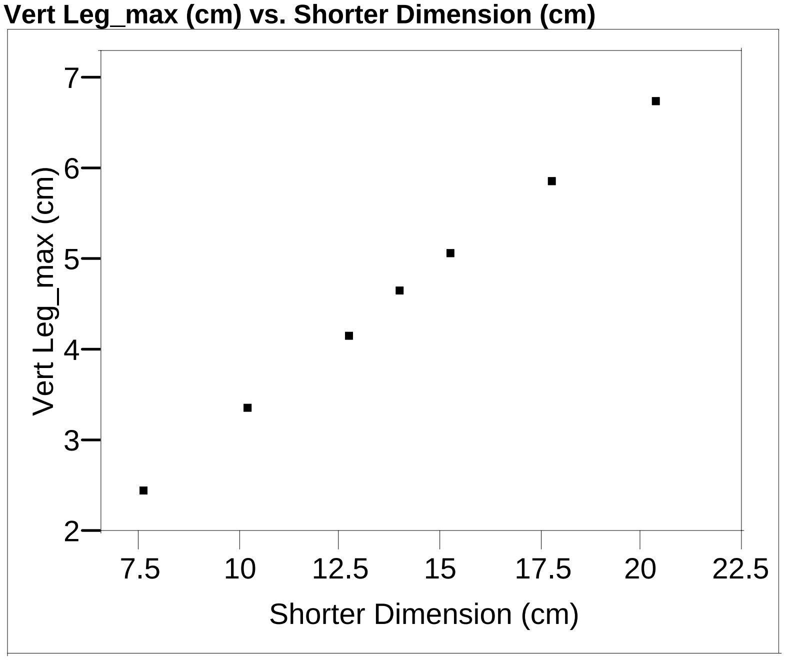
Horiz Leg_max (cm) vs. Longer Dimension (cm)



While there seems to be some relationship; it's not as clearly defined as the relationship between the Horiz. Leg length and the shorter dimension.

So if $h(s) = 0.0257143 + 0.575928s$ (where h is the horizontal leg length that maximizes the area and s is the length of the shorter dimension of the paper) is a good model to predict the horizontal leg length, could we also model the vertical leg length from the shorter-side dimension?

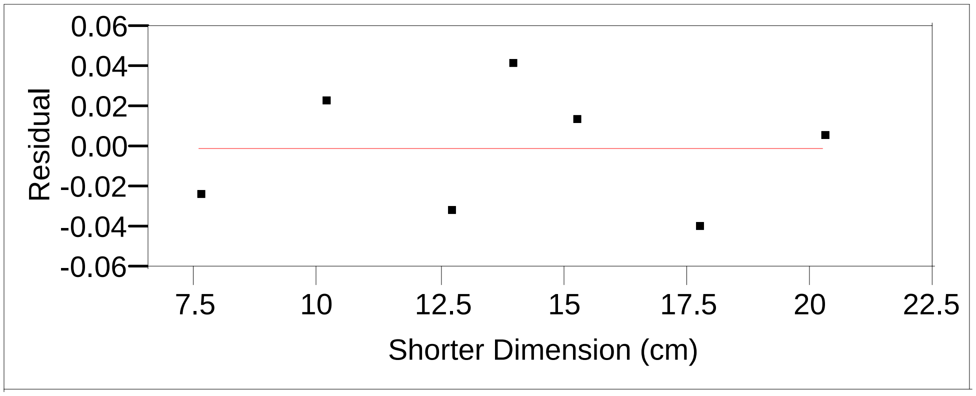
Vert Leg_max (cm) vs. Shorter Dimension (cm)



Looks linear based on the scatterplot; residuals confirm a good fit:

Linear Fit

Vert Leg_max (cm) = -0.041429 + 0.336333 Shorter Dimension (cm)



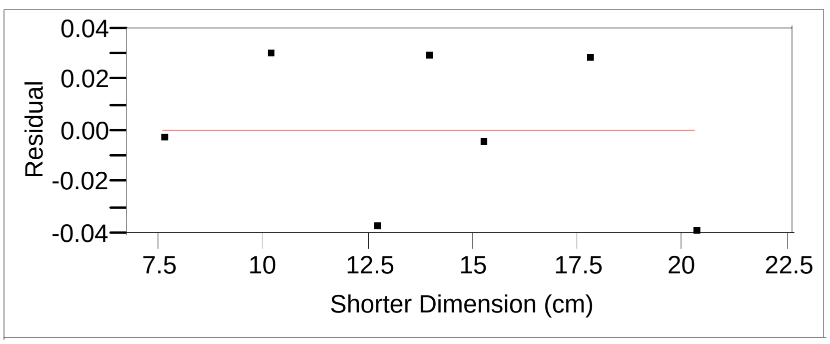
So, we seem to be able to predict the vertical leg length by $v(s) = -0.041429 + 0.336333s$, where v is the vertical leg length that maximizes area and s is the length of the shorter dimension of the paper.

Possible extension:

While the linear models for $h(s)$ and $v(s)$ seem very reasonable; it would perhaps be advantageous for $h(s)$ and $v(s)$ to be "forced" to pass through $(0,0)$, since a shorter-side dimension of 0 cm would necessarily imply triangle legs of length 0. Then our models become $h(s) = 0.5776282s$ and $v(s) = .3335938$ with the following residual plots: (See Appendix 2 for a formula for this regression method.)

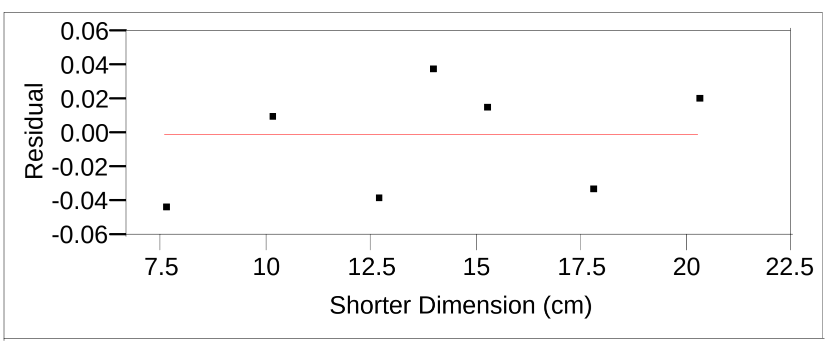
Horiz Leg_max (cm) vs. Shorter Dimension (cm)

Linear Fit
Horiz Leg_max (cm) = 0 + 0.5776282 Shorter Dimension (cm)



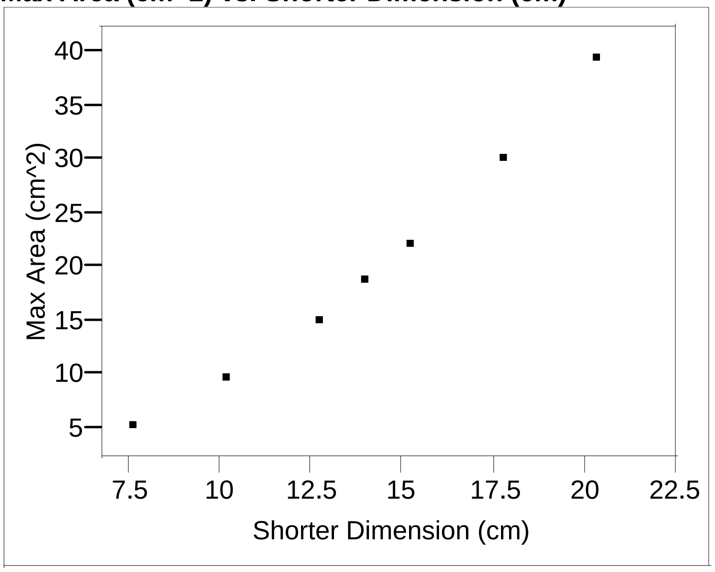
Vert Leg_max (cm) vs. Shorter Dimension (cm)

Linear Fit
Vert Leg_max (cm) = 0 + 0.3335938 Shorter Dimension (cm)



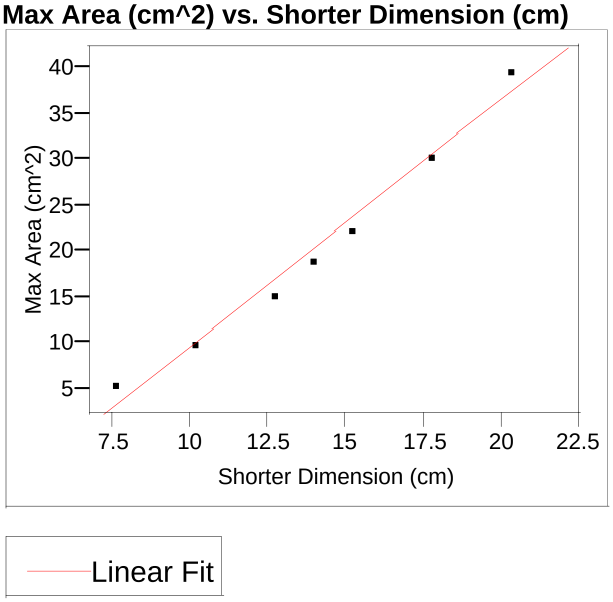
Finally, what if we try to predict the maximum area from the shorter paper dimension?

Max Area (cm^2) vs. Shorter Dimension (cm)



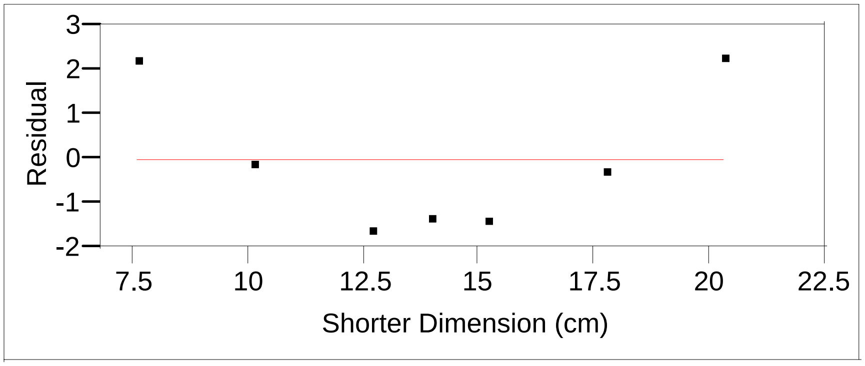
Could a linear model be a good fit? Not when we look at superimposed least-squares line and the residual plot:

Max Area (cm^2) vs. Shorter Dimension (cm)



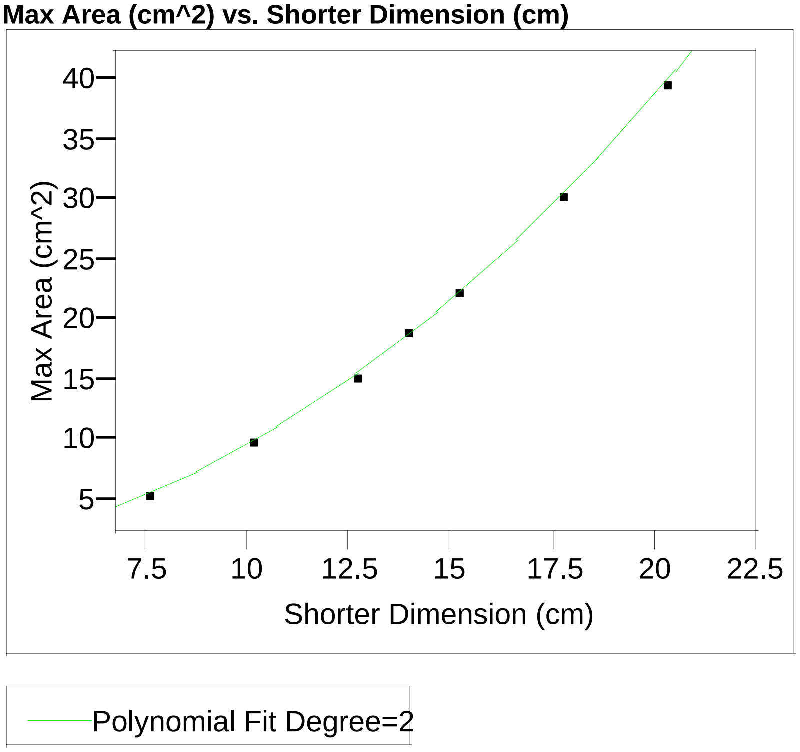
Linear Fit

Max Area (cm^2) = -17.29036 + 2.6948819 Shorter Dimension (cm)



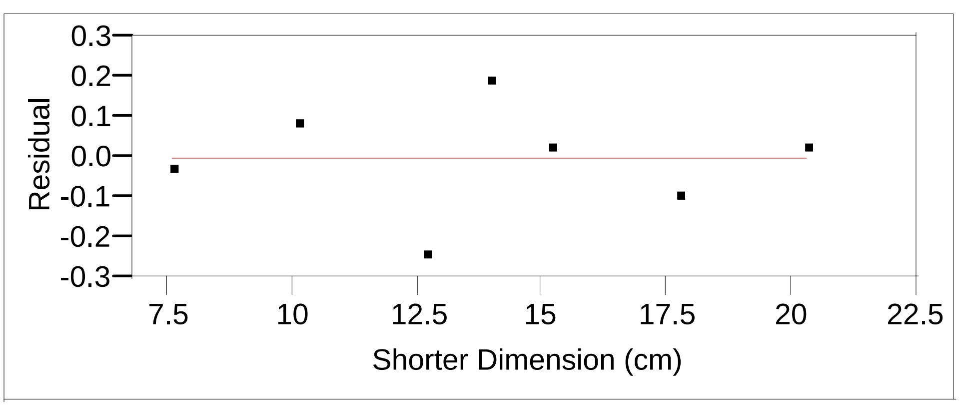
Besides, if each leg is well-modeled by a linear function of the shorter dimension, then it seems reasonable that the area should be modeled by a quadratic function of the shorter dimension:

Max Area (cm^2) vs. Shorter Dimension (cm)



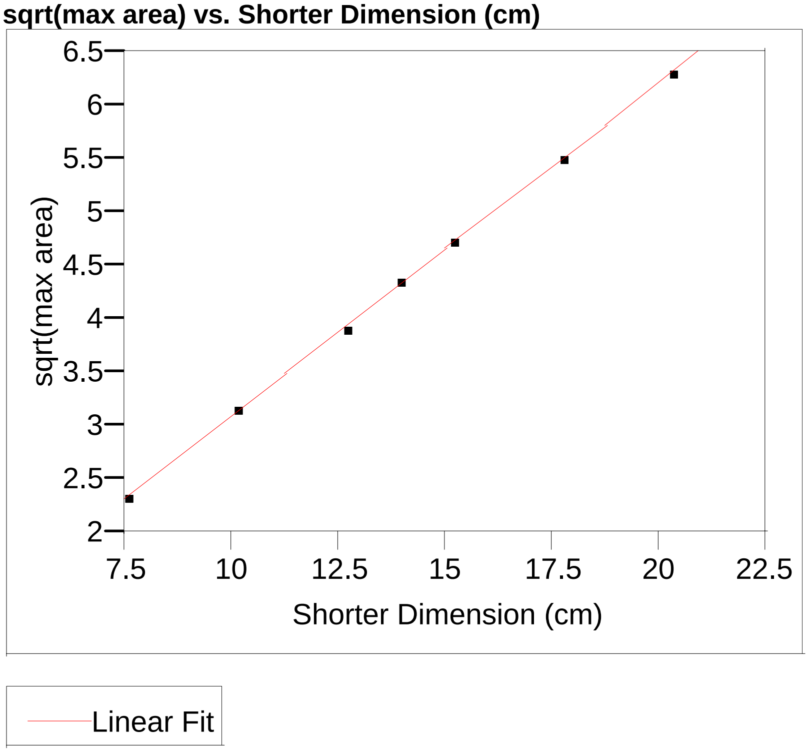
Polynomial Fit Degree=2

Max Area (cm^2) = -18.81004 + 2.6948819 Shorter Dimension (cm) + 0.0942202 (Shorter Dimension (cm)-13.97)^2



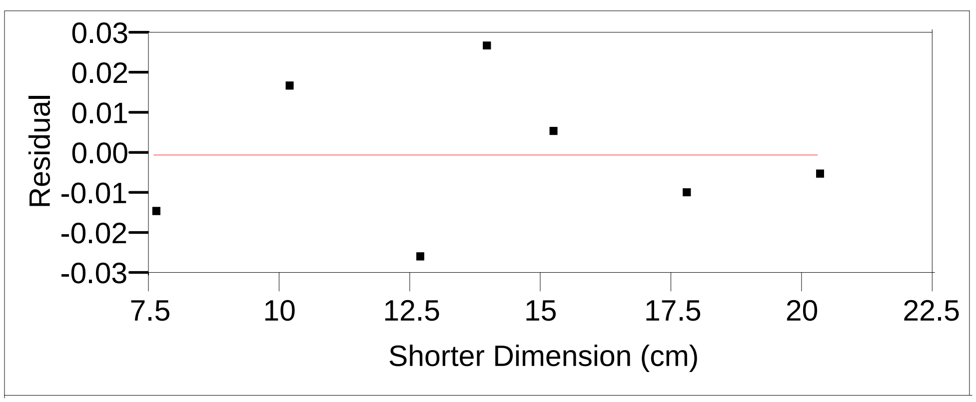
Alternately, we could use re-expression to find a model for the area as a function of shorter dimension. Since we think quadratic is a good model, we can take the square root of the area:

sqrt(max area) vs. Shorter Dimension (cm)



Linear Fit

sqrt(max area) = -0.012407 + 0.3112161 Shorter Dimension (cm)



When we write our model back in terms of the maximum area, we have:

$$\sqrt{a(s)} = -.012407 + .3112161s$$
$$a(s) = (-.012407 + .3112161s)^2$$
$$a(s) = .000154 - .007723s + .096855s^2$$

(where $a(s)$ is the maximum triangle area for a sheet of paper with shorter dimension s).

Again, as a possible extension, we might “force” our linear model through the origin (since it makes sense that a shorter dimension of 0 cm would imply an area of 0). Then our model would be:

$$\sqrt{a(s)} = 0.3103958s$$
$$a(s) = .096346s^2$$

Calculus method

Earlier we had

$$A(h) = \frac{1}{2}h\left(\frac{21.6^2 - h^2}{2(21.6)}\right)$$

as a model to predict the area, A, as a function of horizontal leg length, h. Noticing that the "21.6" is simply the shorter dimension of the sheet of paper (in cm), we can make our solution more general by writing

$$A(h) = \frac{h}{4s}(s^2 - h^2)\,, 0 < h < s$$

where s is the shorter dimension of the sheet of paper. (Notice that, as we observed earlier, the maximum area is dependent only on the shorter dimension of the sheet of paper, not the longer dimension.)

Then we can find the value of h that maximizes A as follows:

$$A'(h) = \frac{1}{4s}\left(\, h\,(-2h) + (s^2 - h^2)\right)$$

$$A'(h) = \frac{1}{4s}(s^2 - 3h^2)$$

Setting $A'(h) = 0$ we have $s^2 = 3h^2$, so $h_{\text{max}} = s/\sqrt{3}$. (An easy check with the 2nd derivative test verifies it is indeed a maximum.)

Alternately, if we had chosen to work with

$$A(v) = \frac{1}{2}v\sqrt{21.6^2 - 2(21.6)v}\,, 0 < v < s/2$$

instead, the algebra is slightly more challenging:

$$A(v) = \frac{1}{2}v\sqrt{s^2 - 2\,s\,v}$$

$$A'(v) = \frac{1}{2}\left(\, v\frac{-2s}{2\sqrt{s^2 - 2sv}} + \sqrt{s^2 - 2sv}\right)$$

$$A'(v) = \frac{1}{2}\left(\frac{-s\,v}{\sqrt{s^2 - 2sv}} + \sqrt{s^2 - 2sv}\right)$$

Then setting $A'(v) = 0$ we have

$$\frac{sv}{\sqrt{s^2 - 2sv}} = \sqrt{s^2 - 2sv}$$

$$sv = s^2 - 2sv$$

$$3sv = s^2$$

$$v_{\text{max}} = \frac{s}{3}$$

Since $h_{\text{max}} = s/\sqrt{3}$ and $v_{\text{max}} = s/3$, it therefore follows that

$$\begin{aligned} A_{\text{max}} &= \frac{1}{2}h_{\text{max}}v_{\text{max}} \\ &= \frac{1}{2}\frac{s}{\sqrt{3}}\frac{s}{3} \\ &= \frac{s^2}{18} \end{aligned}$$

Appendix 1

Excerpts from a sample student handout:

Triangle Problem Writing Assignment

Precalculus, August, 2007

- Write alone, but you may consult with others in *your* group.

- Characteristics of your paper:
  - It should be “self-contained”, meaning that you could mail it to a friend back home who knows the math you know, and s/he would be able to follow your thinking.
  - You can do it all by hand, or type the words and do the rest by hand, or learn MathType and do the equations in a Word doc also.
  - Include your name and the names of your pod-partners, a signature (which indicates that you were academically honest in doing this work), the name of the assignment, the date handed in, and your class block’s letter.

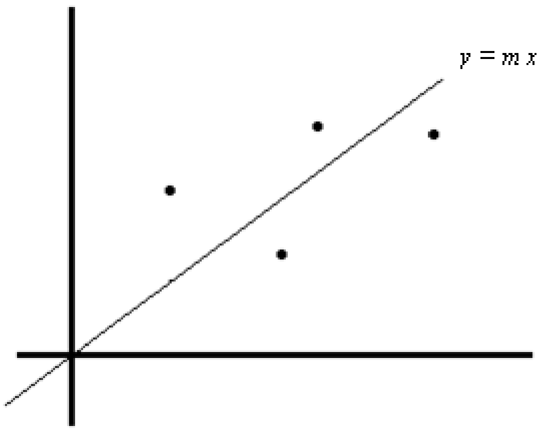
- The paper should contain:
  - A description of the problem. (If you decide to take the description of the problem word-for-word from the handout, you must cite the source.)
  - The “data solution”, which should include (with words wrapped around them, of course):
    - The actual data, including units
    - A scatter plot (with axes labeled with words and a scale)
    - An answer, as an ordered pair and in words
  - The “algebra solution”, which should include:
    - Identifying your variables
    - The work leading up to your model
    - The model
    - A sketch of the model over the domain of reality (and why that *is* the D.of.R)
    - An answer, as an ordered pair and in words
  - A comparison of the two answers

If you want to say more (like, what you got by punching “magic buttons”, or looking at the relationship between base and height of the triangles), feel free

Appendix 2

Linear Regression with a "forced" intercept at 0.

The goal of regression is to minimize the sum of the squares of the residuals. We start with ordered pairs (x_i, y_i) and the linear model $y = m\,x$.



Since a residual is "actual y" – "predicted y," each residual can be written as $y_i - mx_i$. So we set out to minimize the sum of the squares of the residuals, or

$$\begin{aligned} SSR(m) &= \sum_i (y_i - mx_i)^2 \\ &= \sum_i (y_i^2 - 2mx_iy_i + m^2x_i^2) \\ &= \sum_i y_i^2 - 2m\sum_i x_iy_i + m^2\sum_i x_i^2 \end{aligned}$$

To find the value of a that minimizes $SSR(m)$, we take the derivative with respect to m:

$$SSR'(m) = -2\sum_i x_iy_i + 2m\sum_i x_i^2$$

Setting the derivative equal to zero, we have

$$\sum_i x_iy_i = m\sum_i x_i^2$$

therefore

$$m_{\max} = \frac{\sum_i x_iy_i}{\sum_i x_i^2}$$

gives the slope of the line through the origin that minimizes the sum of the squares of the residuals.