

# *Rearranging the Alternating Harmonic Series*

**Dan Teague**  
**NC School of Science and Mathematics**  
**teague@ncssm.edu**

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## *Regrouping Infinite Sums*

We know that the Taylor series for  $\ln(x+1)$  is

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \dots$$

and that this series converges to  $\ln(2)$  when  $x = 1$ . We will begin with the statement that

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots \quad (1)$$

Now, let's rewrite the positive terms in the series for  $\ln(2)$  in the following way,

$$\ln(2) = [2-1] - \frac{1}{2} + \cancel{\frac{1}{3}} - \frac{1}{4} + \cancel{\frac{1}{5}} - \frac{1}{6} + \cancel{\frac{1}{7}} - \frac{1}{8} + \cancel{\frac{1}{9}} - \frac{1}{10} + \dots \quad (2)$$

We haven't changed the order in which the operations have been done, we have simply renamed the positive terms by rewriting each positive term in an equivalent form. Just why we did this remains to be seen, but we still have a series whose sum is  $\ln(2)$ .

Now, divide both sides of Equation (2) by 2. The result is

$$\frac{\ln(2)}{2} = \cancel{\frac{1}{2}} - \frac{1}{4} + \cancel{\frac{1}{6}} - \frac{1}{8} + \cancel{\frac{1}{10}} - \frac{1}{12} + \cancel{\frac{1}{14}} - \frac{1}{16} + \cancel{\frac{1}{18}} - \frac{1}{20} + \dots$$

Removing the parentheses, we have

$$\frac{\ln(2)}{2} = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \frac{1}{9} - \frac{1}{18} - \frac{1}{20} + \dots \quad (3)$$

Compare the terms in the right hand side of Equation (3) to those in the right hand side of Equation (1). Are there any terms in one equation that are not also in the other?

What you will notice is that the two series have **exactly** the same terms. The only difference is in the order in which they are added. Yet one adds to  $\ln(2)$  and the other to  $\frac{1}{2}\ln(2)$ ! If you find this hard to believe, write a short computer program to sum the two series. On the next page, you will find a MathCAD template that does this computation. Both series converge very slowly, but it should be clear that they do not converge to the same values. The associative property of addition is true is not necessarily valid for infinite sums.

$n := 1..1000$

$$S_n := \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$$

$$Z_n := \left[ \sum_{k=1}^n \frac{(-1)}{4 \cdot k - 2} + \frac{(-1)}{4 \cdot k} + \frac{1}{2 \cdot k - 1} \right]$$

$m := 1..10$

$m$	$S_m$
1	1
2	0.5
3	0.8333333333
4	0.5833333333
5	0.7833333333
6	0.6166666667
7	0.7595238095
8	0.6345238095
9	0.7456349206
10	0.6456349206

$(2 \cdot m - 1)$	$(4 \cdot m - 2)$	$(4 \cdot m)$	$Z_m$
1	2	4	0.25
3	6	8	0.2916666667
5	10	12	0.3083333333
7	14	16	0.3172619048
9	18	20	0.3228174603
11	22	24	0.3266053391
13	26	28	0.3293525919
15	30	32	0.3314359252
17	34	36	0.3330699121
19	38	40	0.3343857016

$M := 990..1000$

$M$	$S_M$
990	0.6926423851
991	0.6936514669
992	0.6926434024
993	0.6936504517
994	0.6926444155
995	0.6936494406
996	0.6926454245
997	0.6936484336
998	0.6926464296
999	0.6936474306
1000	0.6926474306

$(2 \cdot M - 1)$	$(4 \cdot M - 2)$	$(4 \cdot M)$	$Z_M$
1979	3958	3960	0.3464473595
1981	3962	3964	0.3464474869
1983	3966	3968	0.346447614
1985	3970	3972	0.3464477408
1987	3974	3976	0.3464478674
1989	3978	3980	0.3464479937
1991	3982	3984	0.3464481198
1993	3986	3988	0.3464482456
1995	3990	3992	0.3464483712
1997	3994	3996	0.3464484965
1999	3998	4000	0.3464486215

In the regrouping above, we took one positive term and two negative terms and repeated the process. The rearranged series converges to  $\frac{1}{2}\ln(2)$ . What other sums can we get?

The two sub-series of negative terms,  $N_n = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \dots$ , and positive terms,  $P_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ , are both divergent series. We can see that  $N_n$  diverges, since  $N_n = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \dots = -\frac{1}{2} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right)$  which is a constant multiplied by the sum of the harmonic series which diverges. Since  $N_n$  diverges, so does  $-N_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots$ , which can be compared to  $P_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ . Each term of  $P_n$  is larger than the corresponding term of  $-N_n$ , so the divergence of  $-N_n$  guarantees the divergence of  $P_n$ . Since each sub-series diverges, we can rearrange the series to converge to any number we choose.

To rearrange the series to sum to  $\pi$ , for example, take enough terms from  $P_n$  to be just larger than  $\pi$ . Say  $P_k < \pi < P_{k+1}$ . Now, add the first term of  $N_n$ .  $P_{k+1} + N_1 < \pi$ . Now add enough additional terms from  $P_n$  until the new sum is just larger than  $\pi$ . Say  $P_m + N_1 < \pi < P_{m+1} + N_1$ . Now, add the second term of  $N_n$ .  $P_{m+1} + N_1 + N_2 < \pi$ . Since each new term is smaller than the previous, the size of the jumps on either side of  $\pi$  get smaller and smaller. If we continue this process, the resulting series converges to  $\pi$ . Unfortunately, we cannot write down the series in a nice compact form like we did with  $\frac{1}{2}\ln(2)$ .

What sums can we get by rearranging consistent clusters of positive and negative terms from the alternating harmonic series. What patterned rearrangement would give us a sum of 0? What would the sum be if we took three positive terms and two negative terms,

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \frac{1}{25} + \frac{1}{27} + \frac{1}{29} + \frac{1}{31} + \frac{1}{33} + \frac{1}{35} + \frac{1}{37} + \frac{1}{39} + \frac{1}{41} + \frac{1}{43} + \frac{1}{45} + \frac{1}{47} + \frac{1}{49} + \frac{1}{51} + \frac{1}{53} + \frac{1}{55} + \frac{1}{57} + \frac{1}{59} + \frac{1}{61} + \frac{1}{63} + \frac{1}{65} + \frac{1}{67} + \frac{1}{69} + \frac{1}{71} + \frac{1}{73} + \frac{1}{75} + \frac{1}{77} + \frac{1}{79} + \frac{1}{81} + \frac{1}{83} + \frac{1}{85} + \frac{1}{87} + \frac{1}{89} + \frac{1}{91} + \frac{1}{93} + \frac{1}{95} + \frac{1}{97} + \frac{1}{99} + \dots$$

or two positives and three negatives,

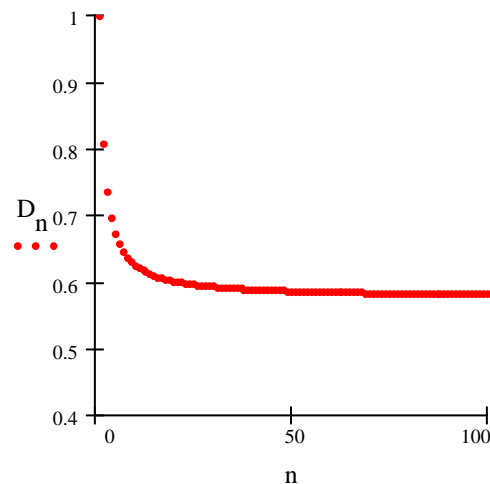
$$S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \frac{1}{25} + \frac{1}{27} + \frac{1}{29} + \frac{1}{31} + \frac{1}{33} + \frac{1}{35} + \frac{1}{37} + \frac{1}{39} + \frac{1}{41} + \frac{1}{43} + \frac{1}{45} + \frac{1}{47} + \frac{1}{49} + \frac{1}{51} + \frac{1}{53} + \frac{1}{55} + \frac{1}{57} + \frac{1}{59} + \frac{1}{61} + \frac{1}{63} + \frac{1}{65} + \frac{1}{67} + \frac{1}{69} + \frac{1}{71} + \frac{1}{73} + \frac{1}{75} + \frac{1}{77} + \frac{1}{79} + \frac{1}{81} + \frac{1}{83} + \frac{1}{85} + \frac{1}{87} + \frac{1}{89} + \frac{1}{91} + \frac{1}{93} + \frac{1}{95} + \frac{1}{97} + \frac{1}{99} + \dots?$$

We can derive the value of the sum of a rearrangement of the alternating harmonic series if the rearrangement is consistent,  $n$  positive terms then  $m$  negative terms. To do this, we need three pieces of information.

- First, we can write the first  $2N$  terms of the harmonic series,  $H_{2N}$ , in terms of the odd and even terms. That is,  $H_{2N} = O_N + E_N$ .
- Second, we need to recognize that  $E_N = \frac{1}{2}H_N$ , since  $E_N = \sum_{n=1}^N \frac{1}{2n}$  and  $H_N = \sum_{n=1}^N \frac{1}{n}$ .
- Third, the difference in the sum of the first  $N$  terms of the harmonic series and  $\ln N$  converges to some constant. This constant is called Euler's number and is often symbolized by  $\gamma$ . This is simply comparing the value of the Riemann summation to the area under the curve. Most of the error is generated in the first 10 terms. To see this, use MathCad to evaluate  $\sum_{n=1}^N \frac{1}{n} - \ln N \approx 0.577$  for  $N = 10, 100, 1000, \text{ and } 1000000$ .

$$D_n := \sum_{k=1}^n \frac{1}{k} - \ln(n)$$

m	$D_m$	M	$D_M$
1	1	990	0.5777206304
2	0.807	991	0.5777201209
3	0.735	992	0.5777196125
4	0.697	993	0.5777191051
5	0.674	994	0.5777185987
6	0.658	995	0.5777180933
7	0.647	996	0.5777175889
8	0.638	997	0.5777170856
9	0.632	998	0.5777165832
10	0.626	999	0.5777160819
		1000	0.5777155816



Using these three pieces of information, we can determine the value to which adding  $n$  positive terms and  $m$  negative terms of the alternating harmonic converges. We need to cleverly add zero twice!

$$S = \lim_{k \rightarrow \infty} (O_{kn} - E_{km})$$

We are grouping the positives (odds) and negatives (evens) in  $k$  groups of  $n$  and  $m$ , respectively. Rewrite  $S$  as

$$S_k = O_{kn} + (E_{kn} - E_{km}) = O_{kn} + E_{kn} - E_{km} = H_{2kn} - \frac{1}{2}H_{kn} - \frac{1}{2}H_{km},$$

using the first two ideas above. Now, compare each of the three harmonic series in the expression above the value of the associated logarithm.

$$S_k = O_{kn} + (E_{kn} - E_{kn}) - E_{km} = O_{kn} + E_{kn} - E_{kn} - E_{km} = H_{2kn} - \frac{1}{2}H_{kn} - \frac{1}{2}H_{km}$$

$$S_k = (H_{2kn} - \ln 2kn) - \frac{1}{2}(H_{kn} - \ln kn) - \frac{1}{2}(H_{km} - \ln km) + \ln 2kn - \frac{1}{2}\ln kn - \frac{1}{2}\ln km.$$

Now, take the limit as  $k \rightarrow \infty$ .

As  $k \rightarrow \infty$ ,  $(H_{2kn} - \ln 2kn) \rightarrow \gamma$ ,  $(H_{kn} - \ln kn) \rightarrow \gamma$ , and  $(H_{km} - \ln km) \rightarrow \gamma$ , so we have

$$S = \gamma - \frac{1}{2}\gamma - \frac{1}{2}\gamma + \lim_{k \rightarrow \infty} \left( \ln 2kn - \frac{1}{2}\ln kn - \frac{1}{2}\ln km \right).$$

But this last limit simplifies to

$$S = \lim_{k \rightarrow \infty} \left( \ln 2 + \ln kn - \frac{1}{2}\ln kn - \frac{1}{2}\ln km \right) = \lim_{k \rightarrow \infty} \left( \ln 2 + \frac{1}{2}\ln kn - \frac{1}{2}\ln km \right) = \lim_{k \rightarrow \infty} \left( \ln 2 + \frac{1}{2}\ln \frac{n}{m} \right)$$

which is just

$$S = \ln 2 + \frac{1}{2}\ln \frac{n}{m}.$$

This formula gives the value  $S = \frac{1}{2}\ln 2$  when  $n = 1$  and  $m = 2$ . Notice that the value of  $S$  is zero if  $n = 1$  and  $m = 4$ .

So when can we rearrange the terms in an infinite series? It turns out that, if a series is only conditionally convergent as is the series for  $\ln(2)$ , we cannot arbitrarily rearrange an infinite number of terms the terms without possibly altering the value of the series. For series that are absolutely convergent, altering the order of the terms does not affect the sum. For series that are conditionally convergent, the terms can be rearranged to form a series that converges to any chosen value, since a conditionally convergent series consists of a divergent sub-series of positive terms and a divergent sub-series of negative terms.

### References:

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