Using Taylor's Theorem and Simpson's Rule to Improve Euler's Method

Euler's Method allows us to generate approximate solutions to ordinary differential equations with an initial condition. In this session, we want to determine error bounds when using Euler's Method, and find improvements that will generate smaller errors without reducing the size of the interval Δx. In the process, we will make several connections via Taylor's Theorem and the Fundamental Theorem of Calculus between Euler's method and the traditional techniques of numerical integration, including Simpson's Rule.

Euler's Method

Given the differential equation $\frac{dy}{dx} = f'(x)$ with initial condition (x_0, y_0), we can use the iteration

$$x_{n+1} = x_n + \Delta x$$
$$y_{n+1} = y_n + f'(x_n, y_n) \cdot \Delta x$$

to generate ordered pairs (x_n, y_n) that approximate points on the graph of the solution to the initial value problem. The computed value y_n is an approximation of the true value $f(x_n)$. How accurate is this estimate y_n and what can we do to make it more accurate with the same interval length Δx?

As an example, consider $f'(x) = 5x^4$ with initial value $(1,1)$, so $f(x) = x^5$. If we let $\Delta x = 0.1$ and use just 1 iteration, we generate the approximation $y_1 = 1.5$ while the true value $f(1.1) = 1.611$. The error (Exact minus Approximate) is 0.111.

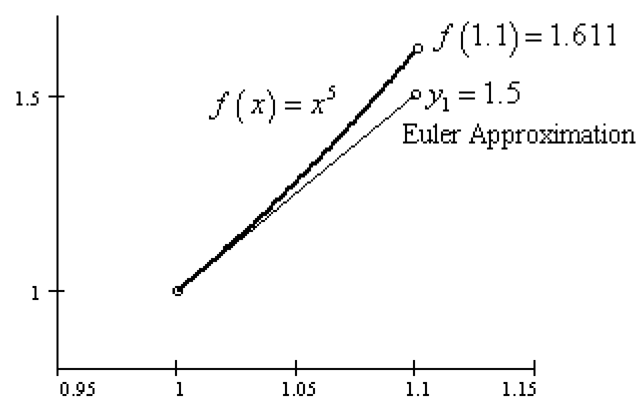


Figure 1: $f(x) = x^5$ and Euler Approximation at $x = 1.1$

Our approximation is too small. Why? What characteristic of *f* determines whether the approximation using Euler's method is too large or too small? By drawing a few graphs, we should see that the curvature of the graph of *f* determines the direction and size of the error in

Euler's method. Consequently, we should expect the error to be related to the second derivative of *f*.

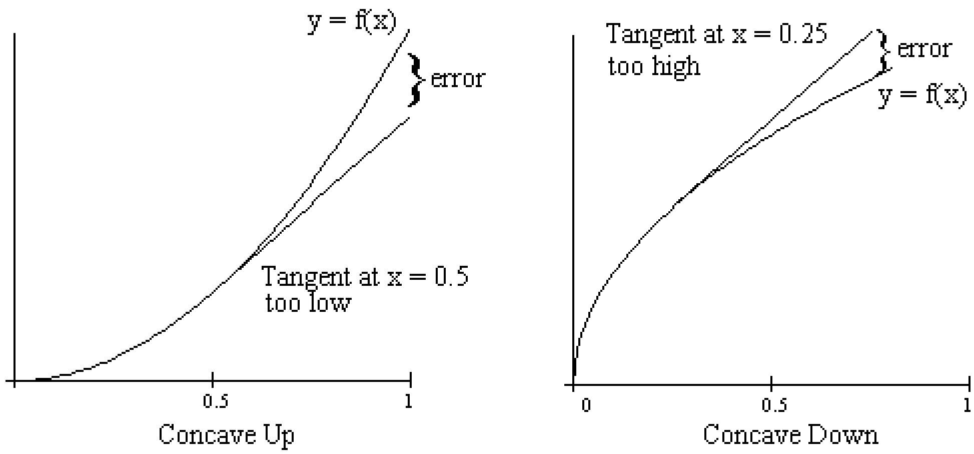


Figure 2: Errors in Euler's Method Tangent Line Approximation

The Fundamental Theorem of Calculus and Euler

We begin by looking at the error in just one iteration of Euler's Method. The Fundamental Theorem of Calculus states that, if f' is continuous on $[x_0, x_1]$ and *f* is any antiderivative of f', then

$$\int_{x_0}^{x_1} f'(x)\,dx = f(x_1) - f(x_0)\,.$$

When solving differential equations, we know that $\frac{dy}{dx} = f'(x)$ and (x_0, y_0) lies on the curve. By the Fundamental Theorem of Calculus we have

$$f(x_1) = f(x_0) + \int_{x_0}^{x_1} f'(x)\,dx\,.$$

We know that $f(x_0) = y_0$ and we want to approximate $f(x_1)$ with y_1. We will rewrite the Fundamental Theorem as

$$y_1 = y_0 + \int_{x_0}^{x_1} f'(x)\,dx\,.$$

(If you are thinking ahead, you will notice that this statement will lead to the iterative equation $y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f'(x)\,dx$.) To compute y_1, we need to evaluate the definite integral $\int_{x_0}^{x_1} f'(x)\,dx$. If there is no error in the integration, there is no error in y_1.

If $\Delta x = x_1 - x_0$ is small, we know that this integral can be approximated using a left-endpoint rectangle. The product $f'(x_0)\cdot\Delta x$ approximates the integral, so the value of $f(x_1)$ can be approximated with

$$y_1 = y_0 + f'(x_0)\cdot\Delta x\,.$$

This is the traditional Euler's method iteration. Consequently, the error in Euler's method is exactly the error in this approximation of the integral. Let's consider how this error should behave. What characteristic of the graph of f' determines the sign and size of the error in using a single rectangle to approximate the integral (we will think of it as an area)?

If the slope of the graph of f' is positive, the rectangle will be too small, and the error (Exact minus Approximate) will be positive. The steeper the slope, the greater the error. But the slope of f' is determined by the size of f'', so we should expect the error in the numerical integration to depend upon the value of f'', just as it was with Euler's method. Let's look at this idea more closely.

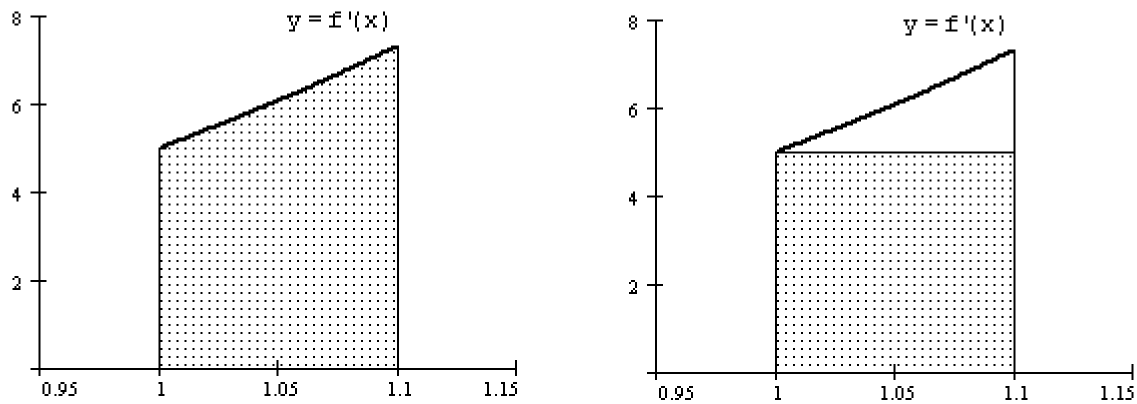


Figure 3: The area under the curve is approximated by the area of the rectangle

The definite integral $\int_1^{1.1} 5x^4 dx = 0.611$. If we use a left-hand rectangle to approximate this area, we have $f'(1)(0.1) = 5(0.1) = 0.5$. The error is 0.111, the same as the error in Euler's Method.

Analysis Using Taylor's Theorem

Now lets consider an analytic approach to this situation. We will change the notation a little to make the algebra easier to read. Our error in Euler's Method in one iteration with interval size h is the same as the error in the numerical integral $\int_a^{a+h} f'(x)\,dx$. By Taylor's Theorem, we know that

$$f'(x) = f'(a) + f''(c)(x-a) \text{ for some } c \in (a, a+h).$$

The value of c required for equality depends upon the value of x at which f is being evaluated. In this analysis, with the interval $(a, a+h)$ small, we consider c to be a constant. This will allow us to compute a good estimate of the exact value of the integral $\int_a^{a+h} f'(x)\,dx$. This estimate of the exact value is

$$E = \int_a^{a+h} f'(x)\,dx = \int_a^{a+h} f'(a) + f''(c)(x-a)\,dx \approx f'(a)h + \frac{f''(c)h^2}{2}.$$

Our approximation using a left-hand rectangle is $A = f'(a)h$, so the error is

$$E - A = f'(a)h + \frac{f''(c)h^2}{2} - f'(a)h = \frac{f''(c)h^2}{2} \text{ for some } c \in (a, a+h).$$

As expected, the error is proportional to the value of the second derivative. In our example with $f'(x) = 5x^4$ on $[1, 1.1]$, $\frac{f''(c)h^2}{2}$ has a maximum value of $\frac{20(1.1)^3(0.1)^2}{2} = 0.1331$, so we know the error in one iteration of Euler's Method will be less than 0.1331. The actual error is 0.11051.

If we repeat the iterations *n* times, we have *n* of these errors hooked together to take us over the interval [*a*, *b*] with $h = \frac{b-a}{n}$, then the accumulated error is approximately

$$E_{Total} \approx n\frac{\left(f''(c)h^2\right)}{2} = \frac{b-a}{h}\frac{\left(f''(c)h^2\right)}{2} = \frac{1}{2}(b-a)\,f''(c)h.$$

The values of *c* in each interval differ, so this is only an approximation of an approximation. However, we notice that decreasing the size of the interval *h* by one-half should cut the accumulated error by one-half. In our example $f'(x) = 5x^4$, if $n = 5$, we have a maximum error of $\frac{1}{2}(b-a)\,f''(c)h = \frac{1}{2}(1.5-1)\,20(1.5)^3(0.1) = 1.6875$. The actual error is 0.97605.

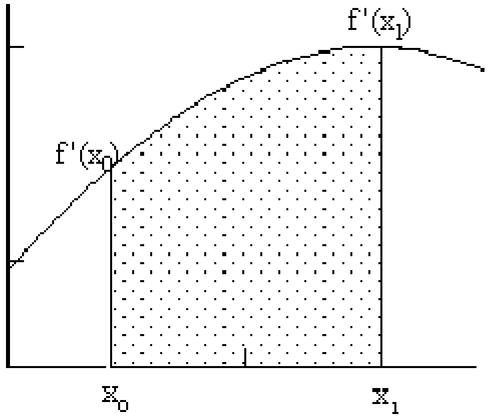
Improving Euler

Now that we know the error in Euler comes from the error in a numerical integration technique, we have several ways of improving Euler's Method. We simply use a better numerical integration technique. We know several: Trapezoid Rule, Midpoint Rule, and Simpson's Rule. All lead to an improved Euler's Method.

Trapezoidal Rule

First, lets consider the Trapezoidal Rule. Here, we find the value of the definite integral $\int_a^{a+h} f'(x)\,dx$ by using

$$\frac{\left(f'(a) + f'(a+h)\right)}{2} \cdot h.$$



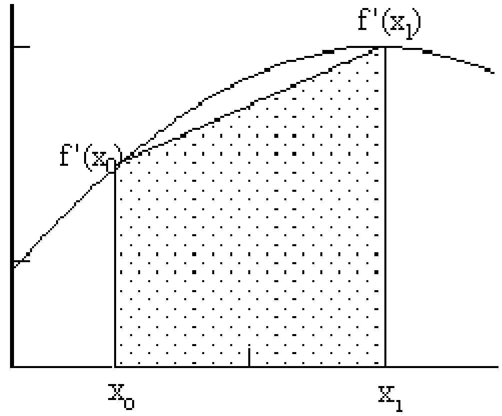


Figure 4: The area under the curve is approximated by the area of the trapezoid

We see that the sign of the error in the trapezoid rule depends upon the concavity of the function f' rather than the slope, so we should expect an error proportional to the second derivative of f', that is, to f'''. Let's see how it works.

By Taylor's Theorem we estimate the exact value of the integral (producing no error in y_1) by considering c constant,

$$E = \int_a^{a+h} f'(x)\,dx \approx \int_a^{a+h} f'(a) + f''(a)(x-a) + \frac{1}{2} f'''(c)(x-a)^2\,dx$$

$$\approx f'(a)h + \frac{f''(a)h^2}{2} + \frac{1}{2}\left(\frac{f'''(c)h^3}{3}\right)$$

We can rewrite the approximation $A = \frac{h}{2}\cdot\left(f'(a) + f'(a+h)\right)$ also using Taylor's Theorem, so,

$$f'(x) = f'(a) + f''(a)(x-a) + \frac{1}{2} f'''(c^*)(x-a)^2 \text{ for some } c^* \in (a, a+h)$$

and

$$f'(a+h) = f'(a) + f''(a)h + \frac{1}{2} f'''(c^*)h^2 \text{ for some } c^* \in (a, a+h).$$

This means that

$$A = \frac{h}{2}\cdot\left(f'(a) + f'(a) + f''(a)h + \frac{1}{2} f'''(c^*)h^2\right) = f'(a)h + \frac{1}{2} f''(a)h^2 + \frac{1}{4} f'''(c^*)h^3.$$

The values of c and c^* are different, but if h is small, we they are approximately the same value. With the simplification $c^* = c$, the error is given by

$$E - A \approx \left[f'(a)h + \frac{f''(a)h^2}{2} + \frac{1}{2}\left(\frac{f'''(c)h^3}{3} \right) \right] - \left[f'(a)h + \frac{1}{2} f''(a)h^2 + \frac{1}{4} f'''(c)h^3 \right]$$
$$= \left(\frac{1}{6} - \frac{1}{4} \right) f'''(c)h^3 = -\frac{1}{12} f'''(c)h^3.$$

Does the sign on the error make sense? If the function f' is concave up, then $f'''(c) > 0$. Also, the Approximate area is too large and $E - A$ is negative. Let's turn this idea into an improvement on Euler's Method.

Huen's Method

The area of a trapezoid is given by the length of the base multiplied by the average height of the two sides, in our case this is $A = \Delta x \cdot \frac{1}{2}\left(f'(x_0) + f'(x_1) \right)$. So $y_1 = y_0 + \int_{x_0}^{x_1} f'(x)\,dx$ can be approximated by

$$y_1 = y_0 + \frac{\left(f'(x_0) + f'(x_1) \right)}{2} \Delta x .$$

Iterating this scheme produces Huen's method

$$y_{n+1} = y_n + \frac{\left(f'(x_n) + f'(x_{n+1}) \right)}{2} \Delta x .$$

The error from Huen's method is very much less than with Euler's method. To see why, look at the defining iteration from a slope of the tangent line point of view. We have

$$y_1 = y_0 + \frac{1}{2}\left(f'(x_0) + f'(x_1) \right) \Delta x .$$

This iteration computes the slope of the tangent at two locations, x_0 and x_1. If the function is concave down (as in Figure 5), we know that Euler's method will overshoot the curve. The next slope computed, $m2 = f'(x_1)$, will therefore be smaller than the previous slope $m1 = f'(x_0)$. By averaging the two slopes, we get a computed value of y_1 that is smaller than that given by Euler's method, and is thus a better approximation. If the function is concave up, just the opposite happens, our computed value of y_1 is larger than the value given by Euler's method.

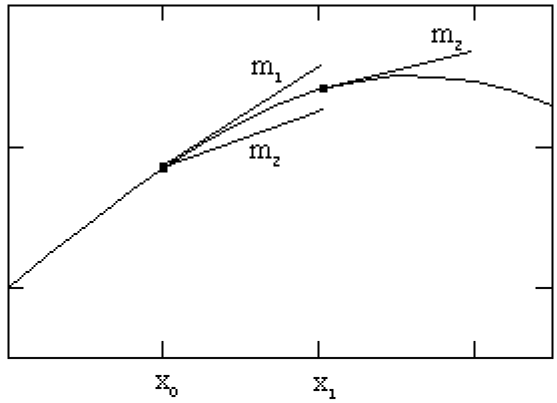


Figure 5: At x_0, move off at a slope that is the average of *m1* and *m2*

The figure below illustrates the improvement of Huen's Method over Euler's Method with one iteration.

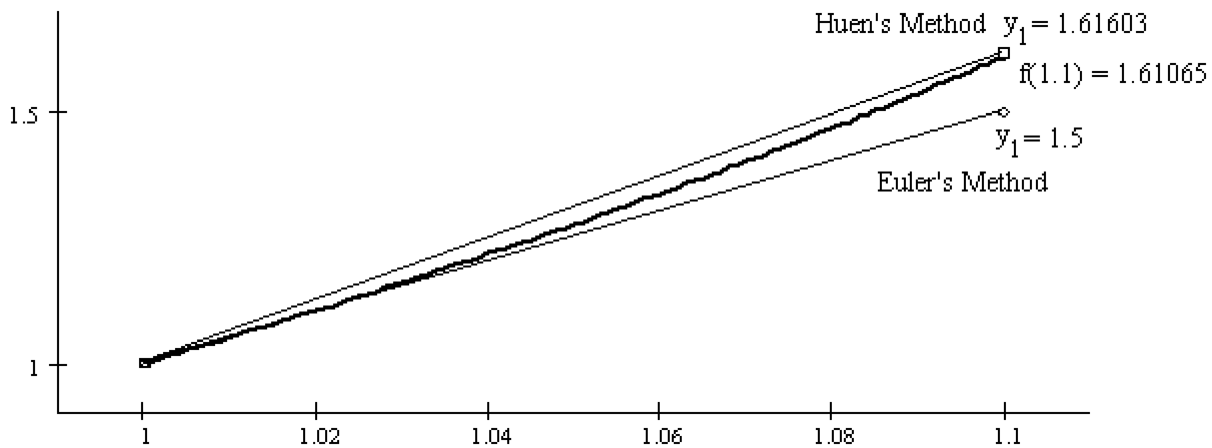


Figure 6: Comparing the Accuracy of Euler's and Huen's Methods

The values are $f(1.1) = 1.61051$ and $y_1 = 1.61603$ with an error of only -0.00552. The maximum value of error bound is

$$-\frac{1}{12} f'''(c) h^3 = -\frac{1}{12}(60)(1.1)^2 (0.1)^3 = -0.00605 .$$

Our observed error is within this bound. If *n* iterations of Huen's Method are used, the combined error is

$$n\left(-\frac{1}{12} f'''(c) h^3\right) = \left(\frac{b-a}{h}\right)\left(-\frac{1}{12} f'''(c) h^3\right) = \frac{(b-a) f'''(c) h^2}{-12} .$$

Cutting the size of the interval *h* in half reduces the accumulated error by one-fourth. Since the error in Huen's Method depends on the third derivative of *f*, we know that Huen's Method will give exact values for quadratic functions.

Differential Equations in *x* and *y*

How does Huen's method work if the derivative is known in terms of both *x* and *y*? If $\frac{dy}{dx} = g(x,y)$, we simply evaluate *g* at each point $\left(x_i, y_i\right)$. This generates the iterative equation

$$y_{n+1} = y_n + \frac{1}{2}\left(g(x_n, y_n) + g(x_{n+1}, y_{n+1})\right)$$

If you look carefully at this equation, you will see a problem. We have defined y_{n+1} in terms of itself! How do we get around using y_{n+1} on the right hand side of the equation? Huen solved the problem by estimating the value of y_{n+1} on the right hand side using Euler's method! That is, Huen defined

$$y^*_{n+1} = y_n + g(x_n, y_n) \cdot \Delta x$$

and then defined y_{n+1} with

$$y_{n+1} = y_n + \frac{\Delta x}{2}\left(g(x_n, y_n) + g(x_{n+1}, y^*_{n+1})\right)$$

Writing a single expression for y_{n+1} in terms of x_n and y_n, we have

$$y_{n+1} = y_n + \frac{1}{2}\left[g(x_n, y_n) + g(x_n + \Delta x, y_n + g(x_n, y_n) \cdot \Delta x)\right] \cdot \Delta x .$$

Midpoint Formula

We can also approximate $\int_a^{a+h} f'(x)\,dx$ by using the midpoint formula $f'\left(a + \frac{h}{2}\right) \cdot h$. What should the error look like and how would this work as an Euler iteration? To determine the error bound, we repeat some of the analysis from the Trapezoidal Rule. We have the same exact value

$$E \approx f'(a)h + \frac{f''(a)h^2}{2} + \frac{1}{2}\left(\frac{f'''(c)h^3}{3}\right).$$

We can also rewrite the approximation $f'\left(a + \frac{h}{2}\right) \cdot h$ using Taylor's Theorem, so,

$$f'\left(a + \frac{h}{2}\right) = f'(a) + f''(a)\frac{h}{2} + \frac{1}{2} f'''\left(c^*\right)\left(\frac{h}{2}\right)^2 \text{ for some } c^* \in (a, a+h)..$$

As before, the values of c and c^* differ, but not by much if *h* is small. The error is

$$E - A \approx \left[f'(a)h + \frac{f''(a)h^2}{2} + \frac{1}{2}\left(\frac{f'''(c)h^3}{3}\right)\right] - \left[f'(a)h + \frac{1}{2} f''(a)h^2 + \frac{1}{8} f'''(c)h^3 \right].$$

This simplifies to

$$E - A \approx \left(\frac{1}{6} - \frac{1}{8}\right) f'''(c) h^3 = \frac{1}{24} f'''(c) h^3 .$$

This error is half that of the Trapezoidal Rule and has the opposite sign. Does the sign of this error make sense?

To implement this as an iteration, we consider

$$y_{n+1} = y_n + f'\left(x_n + \frac{\Delta x}{2}\right) \cdot \Delta x .$$

What is happening in terms of the slopes of the tangent lines? Instead of using the tangent at x_0, we move ahead and compute the slope of the tangent half way through the interval, but draw it at x_0. If f' is concave down, this slope at $x_0 + \frac{h}{2}$ will be smaller than the one at x_0, so the slope will give a better approximation. If f' is concave up, this slope at $x_0 + \frac{h}{2}$ will be larger than the one at x_0.

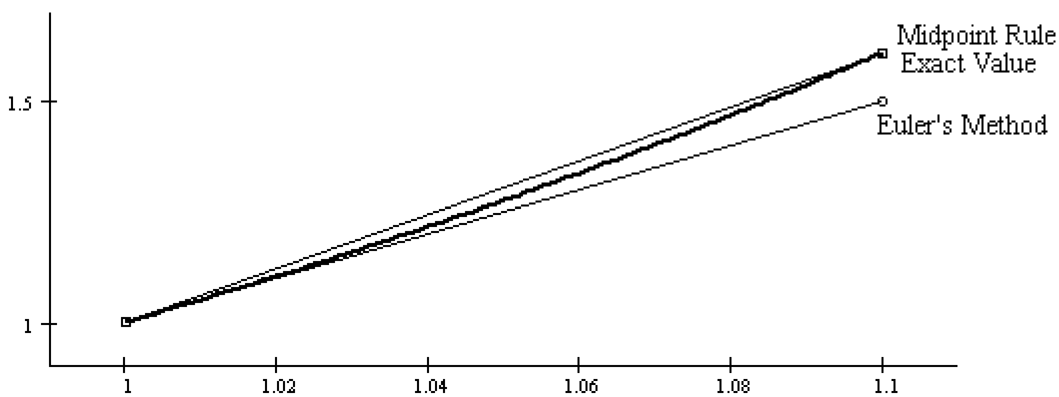


Figure 7: Comparing the Accuracy of Euler's and the Midpoint Methods

The values are $f(1.1) = 1.61051$ and $y_1 = 1.60775$ determined using the midpoint formula. The error is 0.00276, half as large and with the opposite sign as Huen's Method. The maximum value of error bound is

$$\frac{1}{24} f'''(c) h^3 = \frac{1}{24}(60)(1.1)^2 (0.1)^3 = 0.00303 .$$

Simpson's Rule

Since the error in the Trapezoidal Rule (Huen's Method) is opposite in sign and twice the value of that in the Midpoint Rule, we should be able to put the two together and create an even better form of Euler's Method. By averaging the value from Huen's Method and twice that from the Midpoint Rule, we should find a method with an error close to zero. We create

$$A = \frac{1}{3}\left\{\left[\frac{\left(f'(a)+f'(a+h)\right)}{2}(h)\right]+2\left[f'\left(a+\frac{h}{2}\right)(h)\right]\right\}$$

which simplifies to

$$A = \frac{h}{6}\left(f'(a)+4f'\left(a+\frac{h}{2}\right)+f'(a+h)\right).$$

which should be recognized as Simpson's Rule. To determine the error, we need to use more terms in Taylor's Theorem. We have

$$\int_a^{a+h} f'(x)\,dx = \int_a^{a+h} f'(a)+f''(a)(x-a)+\frac{1}{2}f'''(a)(x-a)^2+\frac{1}{6}f^{(4)}(x)(x-a)^3+\frac{1}{24}f^{(5)}(c)(x-a)^4\,dx$$

Evaluating the definite integral with c constant gives us

$$E \approx f'(a)h+\frac{1}{2}f''(a)h^2+\frac{1}{6}f'''(a)h^3+\frac{1}{24}f^{(4)}(a)h^4+\frac{1}{120}f^{(5)}(c)h^5.$$

Using Simpson's Rule, we have the approximation

$$A = \frac{h}{6}\left(f'(a)+4f'\left(a+\frac{h}{2}\right)+f'(a+h)\right).$$

Rewriting each terms gives

$$A = \frac{h}{6}f'(a)+\frac{4h}{6}\left(f'(a)+f''(a)\left(\frac{h}{2}\right)+\frac{f'''(a)}{2}\left(\frac{h}{2}\right)^2+\frac{f^{(4)}(a)}{6}\left(\frac{h}{2}\right)^3+\frac{f^{(5)}(c^*)}{24}\left(\frac{h}{2}\right)^4\right)$$
$$+\frac{h}{6}\left(f'(a)+f''(a)(h)+f'''(a)\frac{(h)^2}{2}+f^{(4)}(a)\frac{(h)^3}{6}+f^{(5)}(c^{**})\frac{(h)^2}{24}\right)$$

and simplifying

$$A = f'(a)h+\frac{1}{2}f''(a)h^2+\frac{1}{6}f'''(a)h^3+\frac{1}{24}f^{(4)}(a)h^4+\frac{5}{576}f^{(5)}(c)h^5.$$

Finally, $E-A=-\frac{1}{2880}f^{(5)}(c)h^5$. So this technique will be correct for 4th degree polynomials.

To implement this technique as an Euler iteration, we define

$$y_{n+1} = y_n+\frac{1}{6}\left(f'(x_n)+4f'\left(x_n+\frac{\Delta x}{2}\right)+f'(x_n+\Delta x)\right)\cdot\Delta x\,.$$

The graph below indicates the accuracy in one iteration. The true value is $f(1.1) = 1.61051$ and $y_1 = 1.6105104167$ using Simpson's Rule, giving an error of only -0.0000004167. Since $f^{(5)}(x) = 120$ the value of the error $-\frac{1}{2880} f^{(5)}(c) h^5$ is constant. We have $-\frac{120}{2880}(0.1)^5 = 0.0000004167$. This is the predicted error correct to 10 decimal places.

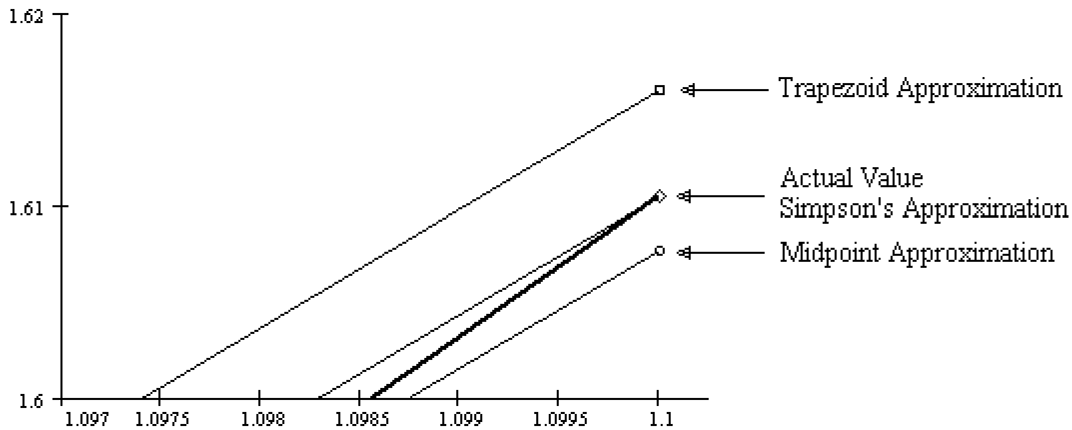


Figure 8: Comparison of Errors for Trapezoid, Midpoint, and Simpson's Approximations

Simpson's Rule is equivalent to the Runga-Kutta Method when the differential equation is a function of *x* only. When $\frac{dy}{dx}$ is a function of both *x* and *y*, the Runga-Kutta Method is a slight modification of Simpson's method. By applying Taylor's Theorem and the Fundamental Theorem of Calculus, we can use Simpson's Rule to improve Euler's Method.

The MathCAD documents following illustrate the relative accuracy for the methods discussed in this paper for solving the differential equation $\frac{dy}{dx} = 5x^4$ with initial conditions $(1,1)$ using $\Delta x = 0.1$. The accuracy of iterations for $N = 1$ (approximating 1.1^5) and $N = 100$ (approximating 11^5) are shown.

As a practical matter, the midpoint method is far better than Euler and very simple to create. There is not much difference in writing

$$y_{n+1} = y_n + f'(x_n)\Delta x$$

and

$$y_{n+1} = y_n + f'\left(x_n + \frac{\Delta x}{2}\right)\Delta x,$$

so this might be the method of choice for simple problems.

When $\frac{dy}{dx}$ is a function of both *x* and *y* the errors are not so easy to analyze and are larger, since additional error is added with each iteration. The midpoint method is a little more difficult to write. Now, instead of

$$y_{n+1} = y_n + f'(x_n, y_n)\Delta x$$

we have

$$y_{n+1} = y_n + f'\left(x_n + \frac{\Delta x}{2}, y_n + \frac{\Delta y}{2}\right)\Delta x .$$

The question is how to find Δy. We know that $\Delta y = y_{n+1} - y_n$ and $y_{n+1} = y_n + f'(x_n, y_n)\Delta x$ by Euler. Then

$$\Delta y = y_n + f'(x_n, y_n)\Delta x - y_n = f'(x_n, y_n)\Delta x .$$

Putting all this together we have

$$y_{n+1} = y_n + f'\left(x_n + \frac{\Delta x}{2}, y_n + \frac{f'(x_n, y_n)\Delta x}{2}\right)\Delta x .$$

The plot below illustrates the improvement for $f'(x,y) = 2xy$ with initial condition $(0,1)$ using $\Delta x = 0.1$ and 10 iterations. The function being approximated is $y = e^{x^2}$ over the interval $[0,1]$.

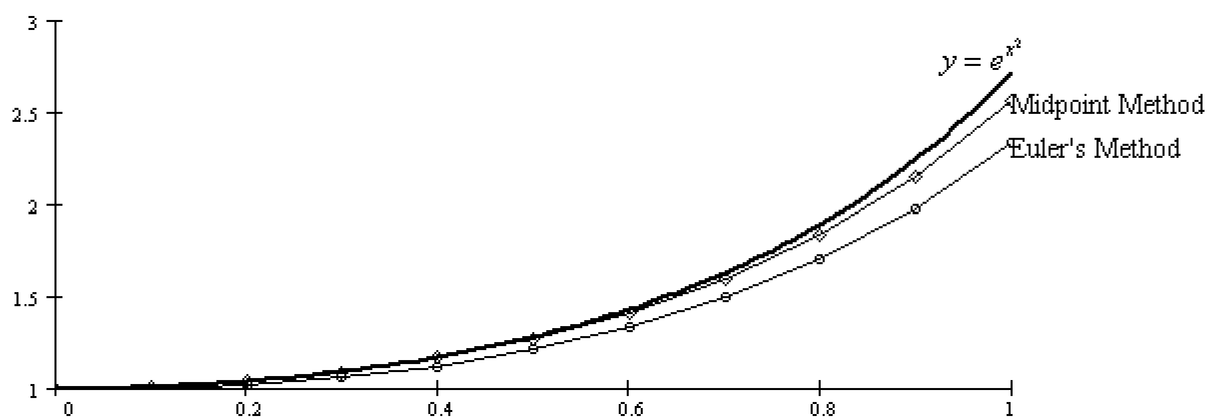


Figure 9: Comparing Euler and Midpoint Methods for $f'(x,y) = 2xy$

MathCAD Comparisons of Error Bounds and Errors

Example 1: The error in 100 iterations with $\Delta x = 0.1$.

$i := 0..N$ $\quad \Delta x := 0.1$ $\quad N \equiv 100$ $\quad g(x) := 5\cdot x^4$ $\quad f(x) := x^5$

Left Endpoint

$ye_{i+1} := ye_i + g(x_i)\cdot\Delta x$ $\qquad f(x_N) - ye_N = 3637.8334999994$

Trapezoid Rule

$$yt_{i+1} := yt_i + \frac{g(x_i) + g(x_{i+1})}{2}\cdot\Delta x \qquad f(x_N) - yt_N = -22.16650$$

Midpoint Rule

$$ym_{i+1} := ym_i + g\left(\frac{x_i + x_{i+1}}{2}\right)\cdot\Delta x \qquad f(x_N) - ym_N = 11.08318749$$

Simpson's Rule

$$ys_{i+1} := ys_i + \frac{\Delta x}{6}\cdot\left(g(x_i) + 4\cdot g\left(\frac{x_i + x_{i+1}}{2}\right) + g(x_{i+1})\right) \qquad f(x_N) - ys_N = -0.0000416672$$

Example 2: One iteration with $\Delta x = 0.1$. Actual error compared to the maximum value of the error bound.

Left Endpoint

$$ye_{i+1} := ye_i + g(x_i)\cdot\Delta x \qquad f(x_N) - ye_N = 0.11051$$

Maximum Error $\quad \frac{20\cdot(x_N)^3\cdot\Delta x\cdot(x_N - x_0)}{2} = 0.1331$

Trapezoid Rule

$$yt_{i+1} := yt_i + \frac{g(x_i) + g(x_{i+1})}{2}\cdot\Delta x \qquad f(x_N) - yt_N = -0.005515$$

Maximum Error $\quad -60\cdot(x_N)^2\cdot\frac{\Delta x^2\cdot(x_N - x_0)}{12} = -0.00605$

Midpoint Rule

$$ym_{i+1} := ym_i + g\left(\frac{x_i + x_{i+1}}{2}\right)\cdot\Delta x \qquad f(x_N) - ym_N = 0.002756875$$

Maximum Error $\quad 60\cdot(x_N)^2\cdot\frac{\Delta x^2\cdot(x_N - x_0)}{24} = 0.003025$

Simpson's Rule

$$ys_{i+1} := ys_i + \frac{\Delta x}{6}\cdot\left(g(x_i) + 4\cdot g\left(\frac{x_i + x_{i+1}}{2}\right) + g(x_{i+1})\right) \qquad f(x_N) - ys_N = -0.0000004167$$

Maximum Error $\quad \frac{1}{2880}\cdot 120\cdot\Delta x^4\cdot(x_N - x_0) = 0.0000004167$

Example 3: Twenty iterations with $\Delta x = 0.1$. Actual Error compared to the maximum value of the error bound.

Left Endpoint

$$ye_{i+1} := ye_i + g(x_i)\cdot\Delta x \qquad f(x_N) - ye_N = 19.5667$$

Maximum Error $$\frac{20\cdot(x_N)^3\cdot\Delta x\cdot(x_N - x_0)}{2} = 54$$

Trapezoid Rule

$$yt_{i+1} := yt_i + \frac{g(x_i) + g(x_{i+1})}{2}\cdot\Delta x \qquad f(x_N) - yt_N = -0.4333$$

Maximum Error $$-60\cdot(x_N)^2\cdot\frac{\Delta x^2\cdot(x_N - x_0)}{12} = -0.9$$

Midpoint Rule

$$ym_{i+1} := ym_i + g\left(\frac{x_i + x_{i+1}}{2}\right)\cdot\Delta x \qquad f(x_N) - ym_N = 0.2166375$$

Maximum Error $$60\cdot(x_N)^2\cdot\frac{\Delta x^2\cdot(x_N - x_0)}{24} = 0.45$$

Simpson's Rule

$$ys_{i+1} := ys_i + \frac{\Delta x}{6}\cdot\left(g(x_i) + 4\cdot g\left(\frac{x_i + x_{i+1}}{2}\right) + g(x_{i+1})\right) \qquad f(x_N) - ys_N = -0.0000083333$$

Maximum Error $$\frac{1}{2880}\cdot 120\cdot\Delta x^4\cdot(x_N - x_0) = 0.0000083333$$

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