Plinko SOLUTIONS

Record the results of your five chips, along with those of your classmates, in the table below.

amount won	$0	$100	$500	$1,000	$10,000
frequency					

- Change these frequencies into <u>probabilities</u>.

amount won	$0	$100	$500	$1,000	$10,000
probability					

- Compare your results to the “true values”:

x	0	100	500	1,000	10,000
$p(x)$	396/1024	33/1024	116/1024	248/1024	231/1024

Why are the simulation results not identical to the “true values”?

The simulation only reflects a few trials. The “true values” indicate the probabilities in the <u>long run</u>; i.e., across a potentially infinite number of trials.

- What happens if you drop the chip from Slot 4? Using the diagram of the Plinko board, see if you can deduce which bounces give which dollar amounts. (The first answer has been provided for you.)

<u># of right bounces</u>		<u>winnings</u>
7	→	$10,000
1, 5, 9	→	$1,000
2, 4, 10, 12	→	$500
3, 11	→	$100
0, 6, 8	→	$0

Plinko, Part II

- Use your simulation data from before to calculate the expected winnings from one Plinko chip.

answers will vary, depending on simulation results

- Use the table of "true values" to calculate the same expected value.

x	0	100	500	1,000	10,000
$p(x)$	396/1024	33/1024	116/1024	248/1024	231/1024

μ = E(x) = Σ x*p(x) = $2557.91

- Do the same for standard deviation.

again, answers will vary depending upon simulation results

for the exact probabilities displayed above,

$\sigma = SD(x) = \sqrt{\Sigma(x-\mu)^2 p(x)}$ = $4,035.66

The table below indicates the probability distribution for Plinko if you drop a chip from Slot **4** rather than Slot 5.

x	0	100	500	1,000	10,000
$p(x)$	355/1024	58/1024	157/1024	256/1024	198/1024

- Calculate the expected value of this probability distribution.

E(x) = $2265.92

- Calculate the standard deviation.

SD(x) = $3806.63

- Which slot wins you more money in the long run: Slot 4 or Slot 5?

Slot 5 (higher expected value)

- Which slot yields less variability: Slot 4 or Slot 5?

Slot 4 (lower standard deviation)

Plinko, Part III

Recall the simulation of bounces on a Plinko board. Consider the random variable *y* defined by

y = the number of 1's in twelve 0/1 simulations

- What are the possible values of *y*? 0, 1, 2, ..., 12

- What type of random variable is *y*: continuous or discrete? discrete

- What is the <u>name</u> of the distribution of *y*? Give the values of *n* and *p*.

  y is BINOMIAL with n = 12 and p = 1/2 = 0.5

- Use the probability distribution of *y*, along with the conversion table for Slot 5 (see below), to construct the probability distribution for your winnings on one Plinko chip. Remember that the table below accounts for reflections off the side walls.

# of "right bounces" (*y*)		winnings (*x*)
6	→	\$10,000
0,4,8,12	→	\$1,000
1,3,9,11	→	\$500
2,10	→	\$100
5,7	→	\$0

x	0	100	500	1,000	10,000
p(*x*)	396/1024	33/1024	116/1024	248/1024	231/1024

Compare your answers to the table provided earlier to see if you are correct.

Plinko, Part IV

In *Plinko*, contestants can earn up to a total of 5 Plinko chips. Let the amounts won from these five chips be X_1, X_2, X_3, X_4, and X_5. Suppose Bob Barker now offers us two ways to play *Plinko*: we can drop 5 chips as usual and take the combined winnings, or we can drop 1 chip and win five times the value of that chip. That is, your total winnings T can either be

i. $T = X_1 + X_2 + X_3 + X_4 + X_5$ (the usual way), or
ii. $T = 5X_1$ (Bob's special offer)

We know from past computation that μ_X = \$2,557.91 and σ_X = \$4,035.66.

- What are the expected payoffs for each of these two options?

i. $E(X_1 + X_2 + X_3 + X_4 + X_5) = E(X_1) + \ldots + E(X_5)$
$= \$2557.91 + \ldots + \$2557.91 = \$12,789.55$
ii. $E(5X_1) = 5E(X_1) = 5(\$2557.91) = \$12,789.55$

Each option has the same expected payoff.

- What are the standard deviations for each of these two options?

i. $Var(X_1 + X_2 + X_3 + X_4 + X_5) = Var(X_1) + \ldots + Var(X_5)$
$= (\$4,035.66)^2 + \ldots + (\$4,035.66)^2 = 81432597.30$,
so $SD(X_1 + X_2 + X_3 + X_4 + X_5) = \$9,024.00$
ii. $SD(5X_1) = 5SD(X_1) = 5(\$4,035.66) = \$20,178.30$

Option ii has significantly higher variability.

- Which option would you take?

There isn't a "right" answer. Both options have equal expected payoff, but option ii has higher variability; i.e., it's riskier. Option ii has a greater potential for very high payoff, but it also has a high probability of \$0 payoff.

The Grand Game SOLUTIONS

A contestant has won $1,000. Two of the remaining products cost <u>more</u> than the target price, and only one costs <u>less</u> than the target price.

- What is the probability the contestant will pick the one remaining product that costs less than the target price?

1/3

- Based on this probability alone, would you risk your $1,000?

Probably not (no pun intended), since the chance of success is less than 50%.

- The contestant wins $10,000 if he picks the right product, and s/he drops to $0 if s/he picks either of the wrong products. Let *x* represent the contestant's winnings, assuming s/he risks $1,000. Fill in the table below.

x	$0	$10,000
$p(x)$	2/3	1/3

- Based on the table, calculate the contestant's <u>expected</u> winnings if s/he risks $1,000.

E(x) = $0*2/3 + $10,000*1/3 = $3,333.33

- Would you risk $1,000 in *The Grand Game* for a chance at $10,000?

Yes, since the expected winnings ($3333.33) exceed the current stakes of $1000.

Let 'Em Roll SOLUTIONS

- If you roll one die one time, what is the probability you roll a car?

$$3/6 = 1/2 \text{ or } .5$$

- If you roll one die 3 times, what is the probability you <u>never</u> roll a car?

$$(1/2)*(1/2)*(1/2) = 1/8 \text{ or } .125$$

- What is the probability that, in 3 rolls of a die, you will eventually roll a car?

$$1-1/8 = 7/8 \text{ or } .875$$

(alternatively, $1/2 + 1/2*1/2 + 1/2*1/2*1/2 = 7/8$ or .875)

- Let x represent the number of dice in *Let 'Em Roll* that eventually show a car. What is the probability distribution of x?

x is a BINOMIAL random variable with $n = 5$ and $p = 7/8$

- What is the probability that all 5 dice eventually show a car? (That is, what is the chance of winning *Let 'Em Roll*?)

$$P(x = 5) = (7/8)^5 = .5129$$

(a better than 50% chance of winning!)

Who Wants to Be a Millionaire? SOLUTIONS

- Without the use of any lifelines, what is the chance the contestant can guess the correct answer to a question?

1/4

- Without the use of any lifelines, what is the chance the contestant can guess two correct answers in a row?

1/4 * 1/4 = 1/16

- Fill in the following chart.

Number of correct guesses in a row	Probability… …as a fraction	…as a decimal
1	1/4	.25
2	1/16	.0625
3	1/64	.015625
4	1/256	.00390625
5	1/1024	.0009765625
6	1/4096	.000244140625

- How many questions does a contestant have to get correct in a row before you believe s/he is not guessing?

Answers will vary by opinion, but students should generally agree that by 4 or 5 in a row, we believe the contestant is not guessing.

Suppose we want to prove that the contestant is <u>not</u> guessing.

- Write the *null hypothesis* and the *alternative hypothesis* in words.

null hypothesis: the contestant is guessing

alternative hypothesis: the contestant is <u>not</u> guessing

- You see a contestant answer 6 questions correctly in a row. <u>Assuming the null hypothesis is true</u>, what is the probability this would occur? (Hint: use your table.)

1/4096, or .000244140625

- In the previous question, you assumed the null hypothesis was true. Based on the probability you gave, do you find this assumption plausible? If not, what's the alternative?

NO. Assuming the contestant is guessing, the chance he would get 6 correct answers in a row is extremely small. The more plausible alternative explanation is that our assumption is wrong, and the contestant is <u>not</u> guessing.

- Let *p* stand for the percent of <u>all</u> *Millionaire* questions this contestant would get right. If the contestant is guessing, what is the value of *p*?

$p = 1/4$

- Write the null and alternative hypotheses formally, in terms of *p*.

H_0: $p = 1/4$
H_a: $p > 1/4$

Master Key SOLUTIONS

There are 5 keys available: the keys for Prize #1, Prize #2, and the car; the Master Key, which opens all 3 locks; and the “dud” key, which opens nothing. Call these keys 1, 2, Car, Master, and Dud.

Suppose a contestant wins one key.

- What is the probability this key opens the lock for the car?

2 keys out of 5 (Car, Master) will work, so 2/5 or 0.4

The contestant tries the key in the lock for Prize #1, but the key doesn’t open the lock.

- Conditional on this information, which of the original 5 keys could the contestant still have?

three: 2, Car, or Dud

- What is the probability the contestant has a key that unlocks a car?

1 key out of 3 will work (Car), so 1/3 or .333

The contestant tries the key in the lock for Prize #2, but the key doesn’t open that lock, either.

- Conditional on this new information, which of the original 5 keys could the contestant still have?

two: Car or Dud

- What is the probability the contestant has a key that unlocks the car?

1 key out of 2 will work (Car), so 1/2 or 0.5

Now, suppose the contestant has won 2 keys.

- How many possible pairs of keys can the contestant choose from the original 5 keys?

5C2 = 10

- Write out all these possible pairs.

1&2, 1&Car, 1&Master, 1&Dud, 2&Car, 2&Master, 2&Dud, Car&Master, Car&Dud, Master&Dud

- What is the probability the contestant has at least one key that will open the lock for the car?

counting above, 7/10

The contestant tries his/her first key in the lock for Prize #1, but the key doesn't open the lock.

- Conditional on this information, which of the original pairs of keys could the contestant still have?

1&2, 1&Car, 1&Dud, 2&Car, 2&Master, 2&Dud, Car&Master, Car&Dud, Master&Dud (every option except 1&Master)

- What is the probability the contestant has a key that unlocks a car?

counting above, 6/9

The contestant tries the same key in the lock for Prize #2, but the key doesn't open that lock, either.

- Conditional on this new information, which of the original pairs of keys could the contestant still have?

1&Car, 1&Dud, 2&Car, 2&Dud, Car&Master, Car&Dud, Master&Dud

- What is the probability the contestant has a key that unlocks the car?

counting above, 5/7

Master Key, Part II

Now we consider the contestant's ability to price the two small prizes. For notational purposes, let p = P(winning a key).

Consider the following questions:

a) What will be the value of p if there is no knowledge of the product price and the contestant is merely guessing?

Out of 2 possible answers, only one is correct so p = 0.5

b) What will be the value of p if the contestant has "perfect" knowledge and always prices items correctly?

The contestant is going to be correct every time so p = 1

(Therefore, it seems reasonable to assume $0.5 \le p \le 1$.)

c) Suppose p = 0.5. What is the probability that the contestant wins no prize? (You first have to consider how many keys s/he will win, then the chances of a key earning a prize.)

The only way to win nothing is either not to win any keys or with one key choose the dud key.

P(not winning any keys) = $\left(\frac{1}{2}\right)^2$

P(winning one key) = 1 – P(winning no keys) – P(winning both keys) = $1 - 0.5^2 - 0.5^2 = 0.5$

P(picking the dud key with one key) = $\frac{1}{5}$

So – P(winning one key and then picking the dud key) = (0.5)(0.2) = 0.1

To answer the question P(winning nothing) = 0.25 + 0.1 = 0.35

d) What is the probability that you win no prize as a function of p?

Consider the following alternatives: (remember, the probability of getting a key = p)

	First product	Second product	Prob.
No key:	No	No	(1-p)(1-p)
1 key:	Yes	No	p(1-p)
OR	No	Yes	(1-p)p
[Total:			2p(1-p)]
2 keys:	Yes	Yes	p^2

Let X = number of keys earned. Let D refer to the dud key, MK refer to the Master key, L to the key that wins the large prize (car), S the key that wins the small prize and M the medium prize.

The only way not to win anything, having won a key is to pick the dud key. $P(D|X = 1) = 0.2$

P(not winning any prize) = $P(\{X=0\} \text{ OR } \{X=1\} \cap D)$ =
$P(X=0) + P(X=1)P(D|X=1) = (1-p)^2 + 2p(1-p)(0.2)$

e) What is the probability that you win a prize but not the car?

f) What is the probability that you win a car?

At this point it is much easier to look at a tree diagram with the calculations as follows:

We have already looked at the options if you win no keys – you don't win a prize!

Suppose you win one key.

P(not winning anything) = P(D) = $\frac{1}{5}$

P(winning a prize *but not the car*) = P(M) + P(S) = $\frac{2}{5}$

P(winning the car) = P(MK) + P(L) = $\frac{2}{5}$

Now suppose you have 2 keys.

P(not winning anything) is obviously 0. You can only pick the dud key once so the other key must open *something*!

The options left are to win a car or not win a car (i.e another prize). There are ${}^{5}C_{2} = 10$ ways to pick 2 keys from the 5 available keys. Of the five keys, 3 of them will NOT unlock the car. In order NOT to win the car, both keys you choose must come from these 3. There are ${}^{3}C_{2} = 3$ ways to do this.

P(not winning the car with 2 keys) = $\frac{{}^{3}C_{2}}{{}^{5}C_{2}} = \frac{3}{10}$

So P(winning the car with 2 keys) = $\frac{7}{10}$

See tree diagram. From this, it is quite easy to calculate P(A), P(B) and P(C).

P(A) = $(1-p)^2(1) + 2p(1-p)(0.2) = \frac{5-8p+3p^2}{5}$

P(B) = $2p(1-p)\left(\frac{2}{5}\right) + p^2\left(\frac{3}{10}\right) = \frac{8p-5p^2}{10}$

P(C) = $2p(1-p)\left(\frac{2}{5}\right) + p^2\left(\frac{7}{10}\right) = \frac{8p-p^2}{10}$

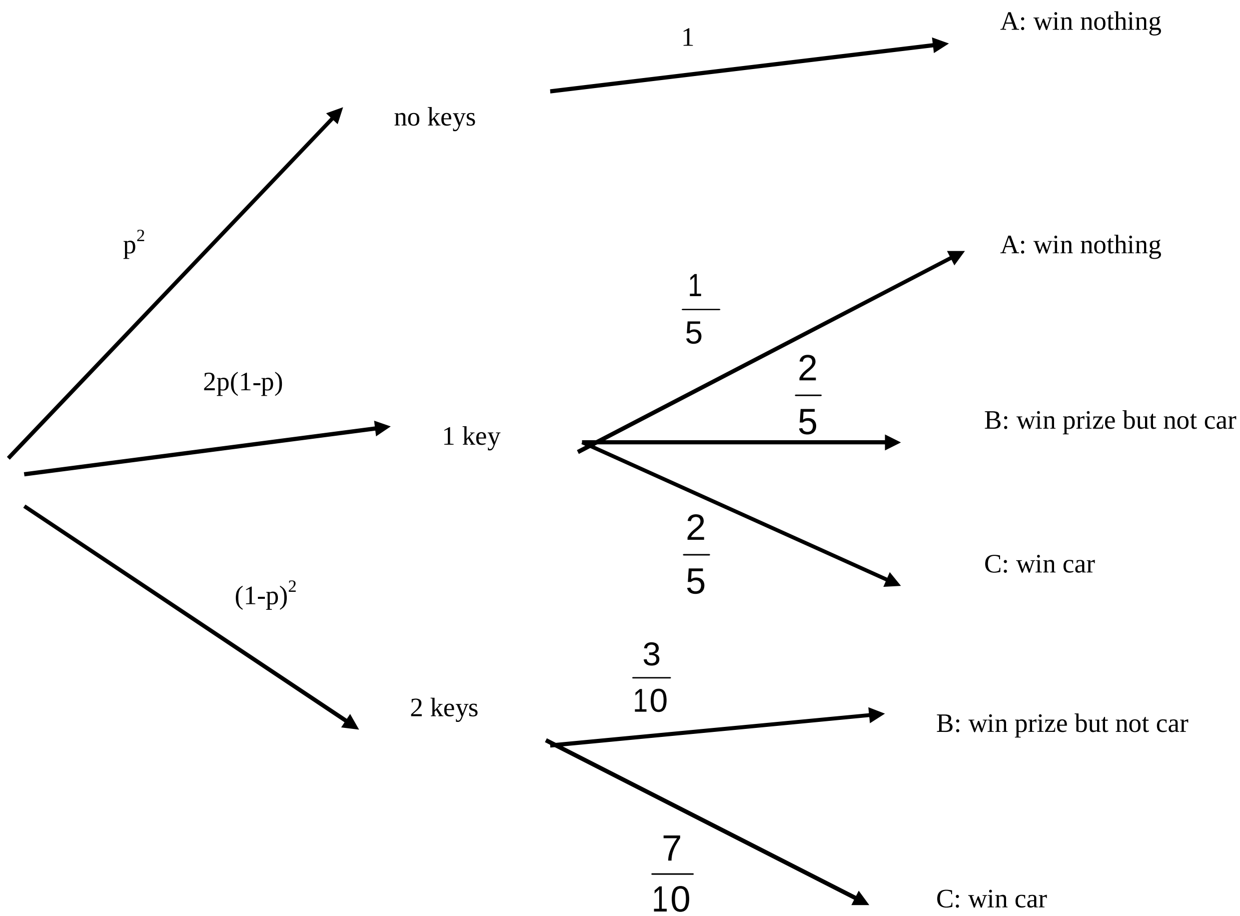
g) Given you won a car, what is the probability that you only have one key?

Using Bayes rule:

$$P(X=1|C) = \frac{P(X=1\cap C)}{P(C)} = \frac{2p(1-p)(\frac{2}{5})}{\frac{8p-p^2}{10}}$$

$$= \frac{8p-8p^2}{8p-p^2}$$

TREE DIAGRAM FOR MASTER KEY



Now consider two contestants, Mr. Pricer and Mr. Guesser who have a probability of getting the product price correct of 1 and 0.5 respectively. To summarize, fill in the following tables and calculate the probabilities of each event for each of our contestants. Does being able to guess product prices well greatly increase the chance of winning a car?

	$.5 \leq p \leq 1$	**Mr. Guesser: $p = 0.5$**	**Mr. Pricer: $p = 1$**
P(A)			
P(B)			
P(C)			
P(X=1|C)			

Extension: Suppose that probability of getting the correct price of the first product is NOT the same as probability of getting the correct price of the second product (i.e., $p_1 \neq p_2$).
Find the probabilities of all the events above in terms of p_1 and p_2.

Spelling Bee SOLUTIONS

Suppose you select 5 of the 30 numbers from the *Spelling Bee* board.

- How many possible groups of 5 numbers could you have? (Hint: Does order matter?)

$$30C5 = \frac{30!}{25!5!} = 142{,}506$$

- Twice in the history of *The Price Is Right*, a contestant has spelled CAR three times with 5 cards: C A R CAR CAR! What is the probability of this amazing event?

$$\frac{11C1 * 11C1 * 6C1 * 2C2}{30C5} = \frac{11 * 11 * 6 * 1}{142{,}506} = .00509$$

- Write out all the different ways you can win *Spelling Bee*. For example, C A R CAR CAR is one way, but so is C C R A C. (Again, keep in mind: Does order matter?)

There are an incredible number of options:

1. two CARs and any three of the 28 letters
2. one CAR and any four of the 28 letters
3. certain sets of five letters: <u>CCCAR</u>, <u>CCAAR</u>, <u>CCARR</u>, <u>CAAAR</u>, <u>CAARR</u>, <u>CARRR</u>

- Find the probability of each of these options.

P(two CARs and any 3 letters) = $\frac{2C2 * 28C3}{30C5} = \frac{1 * 3276}{142,506} = \frac{3276}{142,506}$

P(one CAR and any 4 letters) = $\frac{2C1 * 28C4}{30C5} = \frac{2 * 20,475}{142,506} = \frac{40,950}{142,506}$

P(CCCAR) = P(CAAAR) = $\frac{11C3 * 11C1 * 6C1}{30C5} = \frac{165 * 11 * 6}{142,506} = \frac{10,890}{142,506}$

P(CCAAR) = $\frac{11C2 * 11C2 * 6C1}{30C5} = \frac{55 * 55 * 6}{142,506} = \frac{18,150}{142,506}$

P(CCARR) = P(CAARR) = $\frac{11C2 * 11C1 * 6C2}{30C5} = \frac{55 * 11 * 15}{142,506} = \frac{9075}{142,506}$

P(CARRR) = $\frac{11C1 * 11C1 * 6C3}{30C5} = \frac{11 * 11 * 20}{142,506} = \frac{2420}{142,506}$

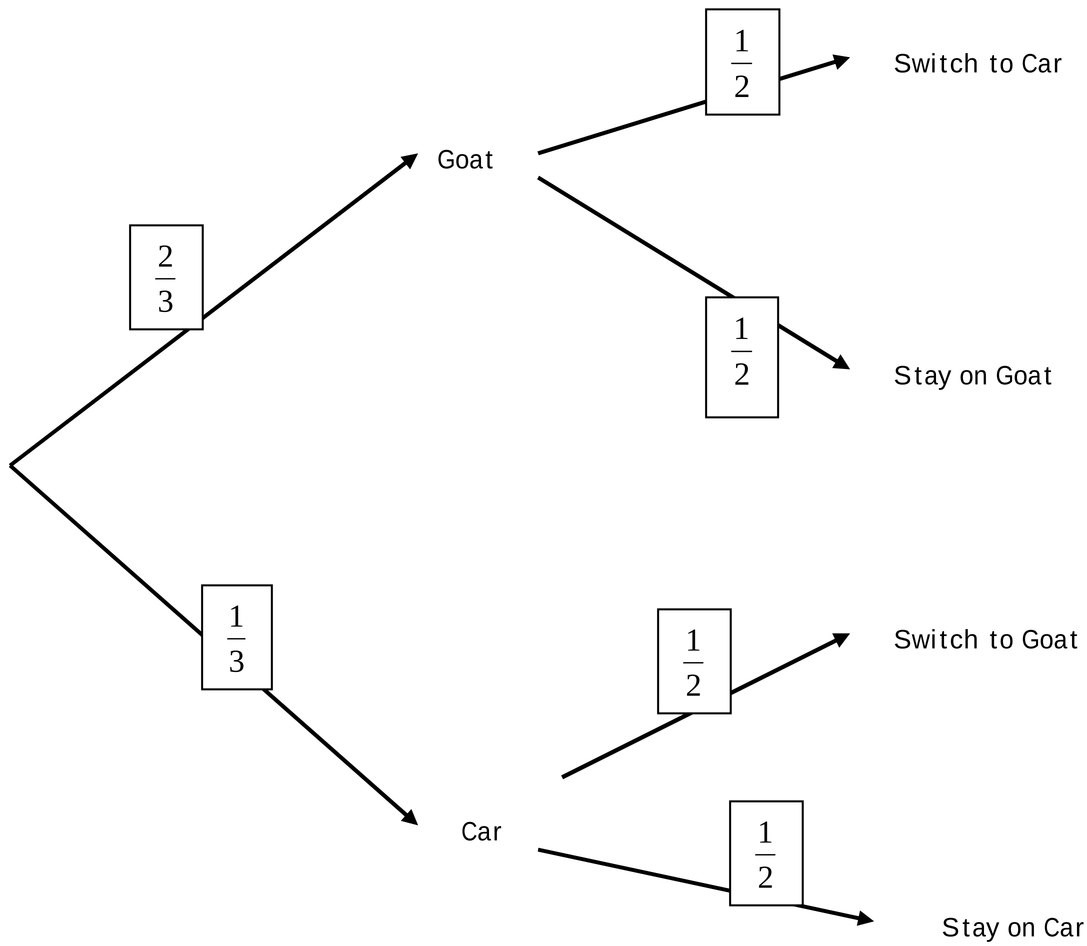
- Add up your answers above to find the probability of winning *Spelling Bee*.

numerator grand total:
3276 + 40,950 + 2*10,890 + 18,150 + 2*9075 + 2420 = 104,726

so, the probability of a win is $\frac{104,726}{142,506} = .7349$

That's right - there is a better than 73% chance of winning *Spelling Bee* if the contestant gets to pick 5 cards from the board.

TREE DIAGRAM FOR THE MONTE HALL PROBLEM



$$P(Car \mid switched) = \frac{P(Car \cap switched)}{P(switched)} = \frac{1/3}{1/2} = \frac{2}{3}$$