

Pass the Candy - An Introduction to Recursive Equations

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Equipment and materials needed for students:

1. Copy of handout, 1 Student Handout for each student.
2. Graphing calculator
3. Paper and pencil for note taking. Colored pencils.
4. Small bags with pieces of individually wrapped bite-size candy. One bag for each student. Each bag should contain an even number of pieces; number of pieces should vary from 2 to 30. You can also use small dice-size manipulatives instead of candy.

Note: This activity can be used in one of two ways:

1. As an introduction to recursion. The students will not be able to write the recursive equations, but they can get an idea of what it means to iterate and they can also see how a recursive system might reach an equilibrium level. After students become familiar with writing recursive equations, you can return to this problem and work through it more carefully.
2. As an introduction to coupled or dependent recursive equations *after* students know how to use SEQUENCE mode on the calculators to produce solutions to problems involving recursive equations.

Activity 1: Pass the Candy Please

Divide the class into groups. Each group should be made up 3 students. If your class size is not evenly divisible by 3, place the "extra" students in a group so that you have no more than 4 students in each group. Arrange the students so that they are seated in a circle.

Each student should be given a tally sheet and a bag of candy. On the tally sheet, have each student record the number of candies in each group member's bag. These values are our initial starting values.

You will decide when an "iteration" occurs. At your instruction ("Pass The Candy!") each student will divide their candy in half and pass half of their candy to the person on his or her left, keeping the remaining half for themselves. Have each student count the number of candies in his or her "new" pile (what they kept plus what their neighbor gave them) and have them record the number of candies for each group member on the tally sheets. At your instruction each student will repeat the iterative process, halving their candy and passing half to the left.

What should we do if a student has an odd number of candies in their pile? For starters, you can have the students keep the extra piece for themselves. For example, if I have 13 pieces of candy, I will keep 7 and pass 6 when I am told to iterate. Continue this iterative process and observe the outcome.

There are three possible outcomes

1. Each person ends up with the same number of candies. An example of initial values that yield this result are [2, 30, 10].
2. Each person in the group doesn't have the same amount as the other members, but each group member's amount doesn't change from step to step. An example of initial values that yield this result are [0, 10, 16].
3. Each person in the group doesn't have the same amount as the other members, but the system gets caught in an infinite loop where the "extra" piece or pieces of candy move(s) from member to member. An example of initial values that yields this result are [20, 16, 4]

These final states can be referred to as equilibriums. In the first outcome described above, the number of candies that each group member receives from their right-hand neighbor is the same as the number of candies they pass to their left-hand neighbor.

After the iterations have settled into the equilibrium state, you can have a member from each group record the following information on the board:

Starting number of candies for each student in their group, final number of candies (the equilibrium value(s)), the type of equilibrium and the number of iterations to reach that state. The students can record the collective information on their Tally Sheets.

If you are using this activity as an introduction to recursion, your students won't be ready to write the recursive equations or use the calculator to iterate. But you can have them think about some questions and have them formulate questions.

Some questions that might arise as we work through this activity are:

Q1: How do the initial starting values affect the final equilibrium value(s)?

Q2: Can we predict the equilibrium value(s) given the initial starting values?

Q3: Will the outcome change if we change our rounding rules? If so, how? For example, if we pass the "extra" piece of candy instead of keeping it for ourselves, will we still reach equilibrium? If so, will the equilibrium value(s) change?

Q4: Can we write an explicit representation (a function) for the number of candies for each group member?

Have students add their own questions to the list above.

As students learn more about recursive equations, they will have the opportunity to explore the answers to some of these questions.

Questions Q1 and Q3 can be answered by using the calculator to model new scenarios. We can explore Question Q2 analytically and Q4 cannot be answered in a course at this level but requires higher-level mathematics taught in a differential equations course.

Note: Some alternate scenarios for exploration are:

1. When a student has an odd number of candies they could contribute a piece to a kitty for the teacher before dividing.
2. When a student has an odd number of candies, they could take a piece from a kitty before dividing.

Using the Java Applet to Explore Various Cases

Located at:

<http://courses.ncssm.edu/math/TCMConf/TCM2004/TCMTalks/Talks2004index.htm>

You can use the Java Applet provided to simulate various scenarios and different rounding rules.

1. Type the initial values for the system in the dialog boxes and hit ENTER.
2. Click on either the Full Step button or Half Step button to see the iterations one-by-one and choose a rounding rule. Ask students to record the initial values, the type of equilibrium and the equilibrium value(s) as they experiment with different scenarios.

Activity 2: Using the TI Calculator to Simulate the Activity

Before starting this activity, students should have experience writing recursive equations and using the calculator to generate numerical values and graphs for recursive equations. This activity can be simulated on the TI - 83 calculator for a 3-person group.

First let's write the recursive equations. Suppose we have 3 students in the group and they are Ursala, Vito and Wanda and suppose Ursala passes to Vito, Vito passes to Wanda and Wanda passes to Ursala.

We will use u_n , v_n and w_n to represent the number of candies at the end of the n th iteration for Ursala, Vito and Wanda, respectively. We will use u_0 , v_0 and w_0 to represent the number of candies for each student at the beginning of the process.

Suppose we have the following initial conditions:

$$u_0 = 2$$

$$v_0 = 30$$

$$w_0 = 10$$

Let's think about writing the recursive equations. The number of candies that Ursala has at the n th iteration will be half of what she had at the time of the previous iteration plus half of what Wanda had at the time of the previous iteration. This leads us to:

$$u_n = \frac{1}{2}u_{n-1} + \frac{1}{2}w_{n-1}$$

For each of the other students, we have similar equations yielding:

$$v_n = \frac{1}{2}v_{n-1} + \frac{1}{2}u_{n-1}$$

$$w_n = \frac{1}{2}w_{n-1} + \frac{1}{2}v_{n-1}$$

Notice these recursive equations depend on one another. Thus we refer to the system as a dependent system of equations. Have the students record these equations on their work sheet. Make sure they include the initial starting values.

Now using sequence mode, type these equations in your Y= and set the starting values as defined above. To see the values of u_n , v_n and w_n , we will use the following settings under TBLSET:

```
TABLE SETUP
TblStart=0
ΔTbl=1
Indent: Auto Ask
Depend: Auto Ask
```

Now looking at the table, we see:

n	$u(n)$	$v(n)$
0	2	30
1	6	16
2	13	11
3	15.5	12
4	15	13.75
5	14.125	14.375
6	13.813	14.25

$u(n)=2$

To see the values for Wanda, scroll to the right using the RIGHT ARROW key. Notice it takes a little while because the calculator is calculating iterates as you scroll. Now you should see:

n	$v(n)$	$w(n)$
0	30	10
1	16	20
2	11	18
3	12	14.5
4	13.75	13.25
5	14.375	13.5
6	14.25	13.938

$w(n)=10$

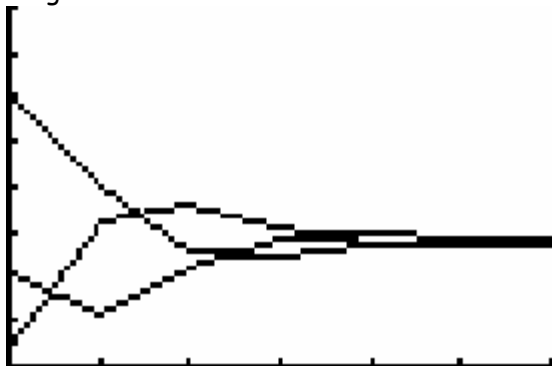
We should notice two things:

- The calculator is not working only with integers
- If we scroll down, we see that there seems to be an equilibrium value of 14.

We can look at a graph of the iterates. Setting the window as follows:

$n\text{Min} = 0$, $n\text{Max} = 6$, $\text{PlotStart} = 1$, $\text{PlotStep} = 1$, $X\text{min} = 0$, $X\text{max} = 8$, $X\text{scl} = 1$,
 $Y\text{min} = 0$, $Y\text{max} = 40$, $Y\text{scl} = 5$

we get



We can use trace on the calculator to look at the initial values and to identify the graph for each student. You may want to trace the graph on paper for each student in a different color to help students identify the amount of candy for each student.

Activity 3: Producing Integer Values with the Calculator

In our class simulation, we know that we want integer values for each iteration since we only pass whole of pieces of candy. How can we get our calculator to do the calculations only with integer values and simulate our odd number rule?

Remembering our rounding rule, when we have an odd number and we divide by 2, we round up for ourselves and round down for our neighbor. Numerically, we want our calculator to "round" $\frac{1}{2}$ of our own portion and take the integer part of $\frac{1}{2}$ of the portion to pass to our neighbor.

We can do this by setting using the *round* and *iPart* functions on the calculator. The *round* function is found by pressing the MATH key, arrow over to NUM and choose #2. The round function takes two parameters. The first parameter is the value to be rounded, and the second parameter is the number of decimal places to be used when rounding. In our case we want to round to the 0th place. The *iPart* function is found by pressing the MATH key, arrow over to NUM and choose #3. The *iPart* function gives the integer part of the number, so *iPart*(10.9) yields 10 and *iPart*(7.0001) yields 7.

Modifying our recursive equations as follows:

$$u(n) = \text{round}(0.5u(n-1), 0) + \text{iPart}(0.5w(n-1))$$

$$v(n) = \text{round}(0.5v(n-1), 0) + \text{iPart}(0.5u(n-1))$$

$$w(n) = \text{round}(0.5w(n-1), 0) + \text{iPart}(0.5v(n-1))$$

Now, when we iterate we will see the following table.

n	$u(n)$	$v(n)$
0	2	30
1	6	16
2	13	11
3	16	12
4	15	14
5	14	14
6	14	14

$n=0$

In checking these values, we see that they are correct. The following table shows all of the iterates.

n	$u(n)$	$v(n)$	$w(n)$
0	2	30	10
1	6	16	20
2	13	11	18
3	16	12	14
4	15	14	13
5	14	14	14
6	14	14	14

We can plot these discrete values by hand or using the calculator. In graphing the iterates, it is useful to have the students use different colors for each graph.

Let's revisit the questions that were posed earlier:

Q1: How do the initial starting values affect the final equilibrium value(s)?

Q2: Can we predict the equilibrium value(s) given the initial starting values?

Q3: Will the outcome change if we change our rounding rules in some way? If so, how? For example, if choose to pass the extra piece when we have an odd number of pieces, will we still reach equilibrium? If so, will the equilibrium value(s) change?

Q4: Can we write an explicit representation (a function) for the number of candies for each group member?

Questions Q1 and Q3 can be answered by using the calculator to model new scenarios. We can explore Question Q2 analytically and Q4 cannot be answered in a course at this level but requires higher-level mathematics taught in a differential equations course.

Now that we know about more about recursive equations, can we answer any of the questions that students posed previously?

Activity 4: Exploring a Solution to Question Q2

To explore the answer to Q2, we should note that some students might have already answered this question using the following method:

One student suggested that that we can predict the equilibrium value(s) by summing the three initial values and then dividing by 3. This solution yields the following:

$\frac{u_0 + v_0 + w_0}{3}$. For our example we have $\frac{2+30+10}{3} = 14$. This correctly yields the equilibrium value for the original scenario. Consider the other two types of equilibriums: $\frac{0+10+16}{3} = 8$, with a remainder of 2. The extra pieces give us the equilibrium values [8, 9, 9]. Finally the initial values [20, 16, 10] yield an equilibrium state where the final values are [14, 13, 13] and the extra piece floats from student t o student. Using our "average", we get $\frac{20+16+4}{3} = 13$ with a remainder of 1.

This method is reasonable for our original scenario.

Activity 5: Exploring a Matrix Solution to Question Q2

We can also explore question Q2, using matrix operations. If your students have studied matrices and can perform matrix multiplication by hand, the answer is attainable and worth exploring.

Note: We will use a simplified rounding rule for dividing odd numbers of candies for our matrix calculations. In the following calculations, we will assume that when we have an odd number of candies, we will round the values down in all cases. For example, if I have 13 candies, the next time I iterate, I will keep 6 for myself and pass 6 to my neighbor and contribute the "extra" piece to a kitty for the teacher.

Here we go:

Let's consider the recursive equations below:

$$u_n = \frac{1}{2}u_{n-1} + \frac{1}{2}w_{n-1}$$

$$v_n = \frac{1}{2}v_{n-1} + \frac{1}{2}u_{n-1}$$

$$w_n = \frac{1}{2}w_{n-1} + \frac{1}{2}v_{n-1}$$

We can re-write these equations as matrix equations as follows:

$$X_n = AX_{n-1}$$

where $X_n = \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix}$, $X_{n-1} = \begin{bmatrix} u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{bmatrix}$, and $A = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$

In performing the matrix multiplication, we have to be careful. Consider the following example:

$$u_0 = 9$$

If our initial values are $v_0 = 20$, we want to perform the following operation

$$w_0 = 35$$

$$X_1 = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 9 \\ 20 \\ 35 \end{bmatrix}$$

where the first row of our resulting matrix is given by $0.5(9) + 0(20) + 0.5(35)$, and each product is rounded down BEFORE we add the three products.

So the first row of X_1 should be $4 + 17 = 21$, not $4.5 + 17.5 = 22$.

If we perform the matrix operation on our calculator, we don't have the control required to perform the rounding *before* we add. This "problem" requires us to perform the matrix multiplication "by hand" and thus delve into the nuts and bolts of how matrix multiplication "really" works!

The result of our multiplication is $X_1 = AX_0 = \begin{bmatrix} 21 \\ 14 \\ 27 \end{bmatrix}$. Now we can use this result to find X_2

since $X_2 = AX_1$. Using matrix multiplication with our "special" rounding, we have

$$X_2 = \begin{bmatrix} 0.5(17) + 0(14) + 0.5(27) \\ 0.5(17) + 0.5(14) + 0(27) \\ 0(17) + 0.5(14) + 0.5(27) \end{bmatrix} \text{ which yields } X_2 = \begin{bmatrix} 21 \\ 15 \\ 20 \end{bmatrix}.$$

We can continue in this manner, computing the values for each iteration.

If we look at this problem analytically and try to predict the equilibrium value given the initial values, we find that our special matrix multiplication presents us with difficulties.

Writing the recursive matrix equations, we have

$$X_1 = AX_0$$

$$X_2 = AX_1 = A(AX_0)$$

Now, we would like to write $X_2 = (AA)X_0 = A^2X_0$, but we can't because our special matrix multiplication is NOT associative. This prevents us from being able to use our matrix equations to write a closed form for the *n*th iterate given the initial values.

Activity 5: Exploring a Matrix Solution Using the Continuous Case

If we abandon our discrete case and instead look at a continuous case, we can explore this matrix solution further and perhaps learn something applicable to the discrete problem.

Consider our recursive matrix equations:

$$X_1 = AX_0$$

$$X_2 = AX_1 = A(AX_0)$$

Now if we use traditional matrix multiplication, which is associate, we get:

$$X_2 = A^2X_0$$

$$X_3 = AX_2 = A^3X_0$$

$$X_4 = AX_3 = A^4X_0$$

•
•
•

$$X_n = AX_{n-1} = A^nX_0$$

So we have found a closed form equation that gives us the values of the *n*th iterates in terms of the initial values. For example given the initial values [2, 30, 10], we can calculate the values at the the10th step without calculating the values for the previous 9 steps.

$$X_{10} = A^{10} \begin{bmatrix} 2 \\ 30 \\ 10 \end{bmatrix}$$

Using your calculator, enter the matrix *A* in a 3 × 3 matrix $\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$ and enter the

initial values in a 3 × 1 matrix, call it *B*. $B = \begin{bmatrix} 2 \\ 30 \\ 10 \end{bmatrix}$.

Now on the HOME screen multiply A^{10} by B. Your result should be

```
([A]^10)*[B]
[[14.015625  ]
 [13.99609375]
 [13.98828125]]
```

We notice that the 10th iterates are very close to 14, our equilibrium value for the discrete system with our original rounding rule. This result is encouraging. Let's take a look at the matrix A^{10} . The values in A^{10} are all very close to $\frac{1}{3}$. In fact if we experiment with A^n ,

for large values of n , we see that $A^n \rightarrow \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. So given our initial values we can

predict the equilibrium value by multiplying A^n , for a large value of n , by our initial values matrix.

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \\ 14 \end{bmatrix}$$

This result agrees with our simpler discrete solution where the student suggested that we add the initial values and divide by 3.

The exploration of this matrix solution leads us to a connection to a class of matrix models

called Markov Chains. The matrix, $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, is called a **stable state matrix**. For more on

Markov Chains see Chapter 8 of *Contemporary Precalculus Through Applications*.

Conclusion

As students work through this problem they are given the opportunity to experiment with both dependent recursive equations and matrix applications. Because of the difficulties we encounter in working with the discrete values, students are made aware of the limitations of our tools. They also have a chance to think about how continuous solutions and discrete solutions may differ and how they might agree. For more practice with dependent recursive systems of equations, see the Great Lakes lesson in *Contemporary Precalculus Through Applications*, Barrett, et al, Glencoe. The Great Lakes lesson provides a wonderful follow-up to this lesson because it involves a continuous solution to a real-world application.

b. Explain how the number of candies changes over time for each group member. Do the values increase or decrease? What happens to the number of candies for the group members in the long run?

3. On the board record the group's starting values, the type of equilibrium and the final equilibrium value(s). Record those values for the entire class in the table below:

Initial Starting Values	Equilibrium Value(s) and Type of Equilibrium	# Iterations to Reach Equilibrium

Some questions to consider:

Q1: How do the initial starting values affect the final equilibrium value(s)?

Q2: Can we predict the equilibrium value(s) given the initial starting values?

Q3: Will the outcome change if we change our rounding rules? If so, how? For example, if we pass the "extra" piece of candy instead of keeping it for ourselves, will we still reach equilibrium? If so, will the equilibrium value(s) change?

Q4: Can we write an explicit representation (a function) for the number of candies for each group member?

Add some of your own questions to the list.

Recursive Equations: Pass The Candy
Student Handout

Name: _____

PART II - This part of the activity should be used *after* you are familiar with writing recursive equations and using the calculator to generate numerical values and graphs for recursive equations.

1. Write the recursive equations for the amount of candy for each group member. Remember to include the initial starting values.

2. Sketch a graph of the number of candies for each person in the group below. Use different colors to represent the number of candies for each group member.

5. Revisit the questions posed earlier:

Q1: How do the initial starting values affect the final equilibrium value(s)?

Q2: Can we predict the equilibrium value(s) given the initial starting values?

Q3: Will the outcome change if we change our rounding rules? If so, how? For example, if we pass the "extra" piece of candy instead of keeping it for ourselves, will we still reach equilibrium? If so, will the equilibrium value(s) change?

Q4: Can we write an explicit representation (a function) for the number of candies for each group member?

Can you answer any of the additional questions that you posed?