Ideas for using Fractals in the classroom.

Fractals are relatively new to the world of mathematics and are richly filled with a variety of mathematics much of which is assessable to students. At a minimum, fractals give an interesting setting to apply and learn some basic mathematics.

What is a Fractal?

[Looking at toolkit functions.]
-Try to have students "build" a fractal dimension. Let them know that you can define a dimension by looking at a ratio of some function applied to the number of boxes to the size of the boxes. Thus

$$Dim = \frac{F(\#\_of\_boxes)}{F(size\_of\_boxes)},$$

determine what function F would make sense in this definition.

[Example of using logarithms.]
-Give students different objects, coastlines, boundaries to calculate their box (fractal) dimension.

How are Fractals constructed?

Fractals are typically constructed by some iteration process. That process can take a geometrical form, a function-composition form, or a function-attractor form.

[This uses Geometry, Sequences, and Series. It also uses Cardinality, Binary and Ternary expansion of numbers, and One-to-One mappings.]
Some typical fractals that can be created geometrically are a Koch snowflake, Cantor's Middle Third Set, and a Sierpinski triangle. Each of these sets has some interesting properties that students can calculate.

- The Koch snowflake has a finite area but an infinite perimeter.
- Sierpinski's Triangle has an infinite perimeter but no area.
- Cantor's Middle Thirds set has zero length but has the same cardinality as the real numbers.

It also has points that are not end points.

[This uses the Iterating Functions/Function Composition.]
Fractals that are created by a function-composition form are typically called L-system fractals. These fractals are created by starting with a set of directions. These directions generally consist of drawing, shifting, and rotating lines. The process then continues by replacing each line drawn in the original set of directions with the complete set of directions. Thus whenever you see part of the object drawn, i.e. a line segment, replace that segment with the whole set of the original rules. You repeat this process at each step. An interesting idea here would be to show that function composition can take the form of just more than "number substitution" but can involve general "rules".

[One can talk about fixed points, periodic points, and strange attractors.]
Fractals that are created by a function-attractor have a few forms, though the idea is the same. By iterating the function over and over again the output will eventually converge to a particular set. Sometimes fractals are produced by looking what the "type of attractor" produced for some function at each point.
An example is the Chaos game which can be found at the web site --
http://www.shodor.org/interactivate/activities/chaosgame/.

[Talking about taking derivatives in the complex plane, maybe using derivatives to show point(s) are attracting/repelling.]
One of the main ideas in showing that these algorithms will produce these fractals is to show that attract to a set or contract to a set. This can be done by showing that the derivative of the function in absolute value is less than one. This in itself is much more interesting when looking at functions in the complex plane. What does it mean to take the derivative of a complex function? This alone can be a very interesting and enlightening discussion. What the derivative means can be as well, though I am not sure I would want to get into that conversation right now.

Complex Derivatives

First note that the derivative of a function f at a point x_0 is defined to be

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Similarly, the derivative for a complex function f at a point z_0 is defined to be

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

These definitions certainly look similar, however, there are some important differences.

1. Under what conditions does the normal derivative not exist?
2. What are the differences between taking a limit on the real line compared to take a limit in the complex plane?
3. Find the derivatives of the following functions, if they exist.
   a. $f(z) = z^2 + c$
   b. $f(z) = \overline{z}$ (the complex conjugate)
   c. $f(x+iy) = x + iy^2$

Three Basic Fractals: Koch Snowflake, Sierpinski's Triangle, and Cantor's Middle Thirds Set

A. The Koch Snowflake is constructed by starting with a triangle and replacing each of the three sides with a _/_ where each piece is one third the length of the original side. To continue making the snowflake, replace each smaller straight side with _/_ where each piece is one third the size of the replaced piece.
    1. Draw a few levels of the Koch Snowflake.
    2. Calculate its area.
    3. Calculate its perimeter.
    4. Calculate its fractal dimension.
    5. Are there any other ways that you can construct the Koch Snowflake?

B. Sierpinski's Triangle can be produced by starting with a filled equilateral triangle (with the base on the bottom). Connect the midpoints of the three sides. This will divide the triangle into four smaller triangles. Remove the middle triangle, leaving three of them. Continue to connect the midpoints of the sides in the filled triangles and remove the middle triangle.
    1. Draw a few levels of Sierpinski's Triangle.
    2. Calculate its area.
    3. Calculate its perimeter.
    4. Calculate its fractal dimension.
    5. Are there any other ways that you can construct the Sierpinski's Triangle?

C. Cantor's Middle Thirds Set is created by taking a line segment and removing it's middle third. This leaves two separated line segments. Next remove the middle third of each of these segments. Continue to repeat this procedure.
    1. Draw a few levels of Cantor's Middle Thirds Set.
    2. Calculate its length.
    3. Calculate its fractal dimension.
    4. Are there any other ways you can construct Cantor's Middle Thirds Set?
    5. Draw a few levels of Cantor's Middle Fifths Set.
    6. Calculate its length.
    7. Calculate its fractal dimension.

Extension: So that the cardinality of Cantor's Middle Thirds set is equal to that of the real line.

Worksheet on Creating and Calculating Fractal Dimension

Fractals are special objects. They are called Fractals because in some sense their dimension is different from objects that we believe we see everyday. We can define a "box" dimension by calculating the ratio of the number of boxes needed to cover an object to the size of the boxes. However we cannot just take the actual number of boxes divided by the size, instead we need to apply some function to these numbers and then take a ratio to have the number consistently make sense.

Consider all of the "Toolkit" functions (linear, quadratic, exponential, trigonometric, etc.) we have discussed this year, and find at least one in which the following definition makes sense.

$$Dim(K) = \frac{F(number\_of\_boxes)}{F(size\_of\_the\_boxes)} = \frac{F(number\_of\_boxes)}{F(magnification\_factor)}$$

1. Start off calculating the dimension of simple objects that you know the dimension of, like a point, line, filled square, etc.
2. Find the dimension of Cantor's Middle thirds set. This is a fractal so what integer dimensions should it be between?
3. Find the dimension of the Koch snowflake. What integer dimensions should this object be between?
4. Find the dimension of Sierpinski's Triangle. What integer dimension should this object be between?
5. Using the maps of several coastlines (of some countries or states), calculate their fractal dimensions.