

## *Regrouping the Alternating Harmonic Series*

In many exercises in this chapter, we have regrouped the terms of an infinite series. On several occasions, we cautioned the student about doing this. We will see the reason for this caution in this problem.

We know that the Taylor series for  $\ln(x+1)$  is at  $x=0$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \dots$$

and that this series converges to  $\ln(2)$  when  $x=1$ . We will begin with the true statement that

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots \quad (1)$$

Now, let's rewrite the positive terms in the series for  $\ln(2)$  in the following way,

$$\ln(2) = [2-1] - \frac{1}{2} + \left[ \frac{2}{3} - \frac{1}{3} \right] - \frac{1}{4} + \left[ \frac{2}{5} - \frac{1}{5} \right] - \frac{1}{6} + \left[ \frac{2}{7} - \frac{1}{7} \right] - \frac{1}{8} + \left[ \frac{2}{9} - \frac{1}{9} \right] - \frac{1}{10} + \dots \quad (2)$$

We haven't changed the order in which the operations have been done, we have simply rewritten each positive term in an equivalent form. Just why we did this remains to be seen, but we still have a series whose sum is  $\ln(2)$ .

Now, divide both sides of Equation (2) by 2. The result is

$$\frac{\ln(2)}{2} = \left[ 1 - \frac{1}{2} \right] - \frac{1}{4} + \left[ \frac{1}{3} - \frac{1}{6} \right] - \frac{1}{8} + \left[ \frac{1}{5} - \frac{1}{10} \right] - \frac{1}{12} + \left[ \frac{1}{7} - \frac{1}{14} \right] - \frac{1}{16} + \left[ \frac{1}{9} - \frac{1}{18} \right] - \frac{1}{20} + \dots$$

Removing the parentheses, we have

$$\frac{\ln(2)}{2} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} + \frac{1}{12} - \frac{1}{7} + \frac{1}{14} - \frac{1}{16} + \frac{1}{9} - \frac{1}{18} + \frac{1}{20} - \frac{1}{20} + \dots \quad (3)$$

Compare the terms in Equation (3) to those in Equation (1). Are there any terms in one equation that are not also in the other? What you will notice is that the two series have **exactly** the same terms. The only difference is in the order in which they are added. Yet one adds to  $\ln(2)$  and the other to  $\frac{1}{2}\ln(2)$ ! If you find this hard to believe, write a short computer program to sum the two series. Both series converge very slowly, but it should be clear that they do not converge to the same values. The associative property of addition is not necessarily valid for infinite sums.

In the regrouping above, we took one positive term and two negative terms and repeated the process. What would the sum be if we took two positive terms and two negative terms,

$$S = \left[1 + \frac{1}{3}\right] + \left[-\frac{1}{2} - \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{7}\right] + \left[-\frac{1}{6} - \frac{1}{8}\right] + \dots$$

or three positives and two negatives,

$$S = \left[1 + \frac{1}{3} + \frac{1}{5}\right] + \left[-\frac{1}{2} - \frac{1}{4}\right] + \left[\frac{1}{7} + \frac{1}{9} + \frac{1}{11}\right] + \left[-\frac{1}{6} - \frac{1}{8}\right] + \dots?$$

We can derive the value of the sum of a rearrangement of the alternating harmonic series if the rearrangement is consistent,  $n$  positive terms then  $m$  negative terms. To do this, we need three pieces of information.

- First, we can write the first  $2N$  terms of the harmonic series,  $H_{2N}$ , in terms of the odd and even terms. That is,  $H_{2N} = O_N + E_N$ .
- Second, we need to recognize that  $E_N = \frac{1}{2}H_N$ , since  $E_N = \sum_{n=1}^N \frac{1}{2n}$  and  $H_N = \sum_{n=1}^N \frac{1}{n}$ .
- Third, the difference in the sum of the first  $N$  terms of the harmonic series and  $\ln(N)$  converges to some constant. This constant is called Euler's number and is often symbolized by  $\gamma$ . This is simply comparing the value of the Riemann summation to the area under the curve. Most of the error is generated in the first 10 terms. To see this, use software to evaluate  $\sum_{n=1}^N \frac{1}{n} - \ln(N) \doteq 0.577$  for  $N = 10, 100, 1000,$  and  $1,000,000$ .

Using these three pieces of information, we can determine the value to which adding  $n$  positive terms and  $m$  negative terms of the alternating harmonic converges. We need to cleverly add zero twice!

$$S = \lim_{k \rightarrow \infty} (O_{kn} - E_{km}).$$

We are grouping the positives (odds) and negatives (evens) in  $k$  groups of  $n$  and  $m$ , respectively. Rewrite  $S$  as

$$S_k = O_{kn} + (E_{kn} - E_{kn}) - E_{km} = (O_{kn} + E_{kn}) - E_{kn} - E_{km} = H_{2kn} - \frac{1}{2}H_{kn} - \frac{1}{2}H_{km},$$

using the first two ideas above. Now, compare each of the three harmonic series in the expression above the value of the associated logarithm.

$$S_k = O_{kn} + (E_{kn} - E_{kn}) - E_{km} = (O_{kn} + E_{kn}) - E_{kn} - E_{km} = H_{2kn} - \frac{1}{2}H_{kn} - \frac{1}{2}H_{km}$$

$$S_k = (H_{2kn} - \ln(2kn)) - \frac{1}{2}(H_{kn} - \ln(kn)) - \frac{1}{2}(H_{km} - \ln(km)) + \ln(2kn) - \frac{1}{2}\ln(kn) - \frac{1}{2}\ln(km)$$

Now, take the limit as  $k \rightarrow \infty$ .

As  $k \rightarrow \infty$ ,  $(H_{2kn} - \ln(2kn)) \rightarrow \mathbf{g}$ ,  $(H_{kn} - \ln(kn)) \rightarrow \mathbf{g}$ , and  $(H_{km} - \ln(km)) \rightarrow \mathbf{g}$ , so we have

$$S = \mathbf{g} - \frac{1}{2}\mathbf{g} - \frac{1}{2}\mathbf{g} + \lim_{k \rightarrow \infty} \left[ \ln(2kn) - \frac{1}{2}\ln(kn) - \frac{1}{2}\ln(km) \right].$$

But this last limit simplifies to

$$S = \lim_{k \rightarrow \infty} \left[ \ln(2) + \ln(kn) - \frac{1}{2}\ln(kn) - \frac{1}{2}\ln(km) \right] = \lim_{k \rightarrow \infty} \left[ \ln(2) + \frac{1}{2}\ln(kn) - \frac{1}{2}\ln(km) \right] = \lim_{k \rightarrow \infty} \left[ \ln(2) + \frac{1}{2}\ln\left(\frac{n}{m}\right) \right]$$

which is just

$$S = \ln(2) + \frac{1}{2}\ln\left(\frac{n}{m}\right).$$

This formula gives the value  $S = \frac{1}{2}\ln(2)$  when  $n = 1$  and  $m = 2$ . The sum is  $\ln(2)$  when both  $n$  and  $m$  are 2, and the sum is  $\ln(2) + \frac{1}{2}\ln\left(\frac{3}{2}\right)$  when  $n = 3$  and  $m = 2$ .

Notice that the value of  $S$  is zero if  $n = 1$  and  $m = 4$ .

So when can we rearrange the terms in an series? It turns out that, if a series is only conditionally convergent as is the series for  $\ln(2)$ , we cannot arbitrarily rearrange the terms without possibly altering the value of the series. For series that are absolutely convergent, altering the order of the terms does not affect the sum. The series in this chapter for which we have rearranged the terms were all absolutely convergent, so everything worked out nicely.