

Calculus and Factorials

Calculus students are familiar with the factorial and know $n!$ as the sum of the first n in positive integers, that is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. We can also define factorials recursively with $0! = 1$ and $n! = n \cdot (n-1)!$ to achieve the same result. A more formal representation is

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & \text{if } n > 0 \end{cases} \text{ and } n \in \mathbb{Z}^+.$$

This definition is often the student's first experience with recursive definitions. Students often wonder why $0! = 1$, but are not surprised when they type $8!$ into their TI-83 the value 40320 show on the screen. They are always quite surprised to see 119292.462 on the screen when they type in $8.5!$. Why would anyone want to know $8.5!$? One reason is that the probability density function for what is known as the t -distribution is

$$P(x) = \frac{1}{\sqrt{n\pi} \left(\frac{n!}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

where n is a positive integer and is called the degrees of freedom. The important part of this is that both $\left(\frac{n}{2}\right)!$ and $\left(\frac{n-1}{2}\right)!$ are terms in the function. Both cannot be integers! Any calculator able to handle t -tests must be able to compute numbers like $8.5!$. To understand what $8.5!$ is and how it is computed requires calculus. Fortunately for us, we have been studying calculus all year, so we should be able to figure this out.

Factorials Revisited

Consider the improper integral $\int_0^{\infty} x^n e^{-x} dx$. The value of the integral depends on the value of n . So we can define a function

$$F(n) = \int_0^{\infty} x^n e^{-x} dx.$$

To evaluate the integral, use integration by parts. So

$$\begin{aligned} u &= x^n & dv &= e^{-x} dx \\ du &= n x^{n-1} dx & v &= -e^{-x} \end{aligned}$$

and

$$\int_0^{\infty} x^n e^{-x} dx = \lim_{k \rightarrow \infty} \left(-x^n e^{-x}\right) \Big|_0^k + n \int_0^{\infty} x^{n-1} e^{-x} dx = 0 + n \int_0^{\infty} x^{n-1} e^{-x} dx$$

This means that $F(n) = nF(n-1)$. That's the basic definition of the factorial. If we can also show that $F(0) = 1$, then the function defined by our integral will indeed be the factorial function we know and love and we can use it to determine $n!$ when n is not an integer.

Is $F(0) = \int_0^{\infty} x^0 e^{-x} dx = 1$? The integral $\int_0^{\infty} e^{-x} dx$ is a standard improper integral and students can easily show that $\int_0^{\infty} e^{-x} dx = 1$. The function $F(n) = \int_0^{\infty} x^n e^{-x} dx$ is the factorial function for all positive integers n . (This also gives another explanation for why $0! = 1$.) Use your calculator to evaluate $\int_0^{\infty} x^n e^{-x} dx$ for $n = 3, 4, 5, 6, \dots$. You can use 100 for the upper limit of integration. This factorial function is a variation of the the gamma function $\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$ which has important applications in statistics.

So what about $8.5!$? Using the numerical integration feature of your calculator to evaluate $\int_0^{\infty} x^n e^{-x} dx$ for $n = 0.5, 1.5, 2.5, 3.5, \dots$. You should notice that $3.5!$ fits nicely between $3!$ and $4!$. You should also notice that your calculator can evaluate $8.5!$ much more quickly than $\int_0^{\infty} x^{8.5} e^{-x} dx$, so it must have some other way to do this.

When we found that $\int_0^{\infty} x^n e^{-x} dx = n \int_0^{\infty} x^{n-1} e^{-x} dx$, or $F(n) = nF(n-1)$, we did not require n to be an integer. This result is true for any real number n . We only required n to be a non-negative integer to match the value with our old friend $n!$. So it is still true that $8.5! = (8.5)7.5! = (8.5)(7.5)6.5! = (8.5)(7.5)(6.5)(5.5)(4.5)(3.5)(2.5)(1.5).5!$. So, all that is required is to find a new initial value. This time instead of $0!$, we need $0.5!$. This integral is a little more difficult than those done earlier.

To evaluate $\int_0^{\infty} \sqrt{x} e^{-x} dx$ we make the substitution $\sqrt{x} = y$, so $x = y^2$ and $dx = 2y dy$.

With this substitution, we have $\int_0^{\infty} \sqrt{x} e^{-x} dx = \int_0^{\infty} 2y^2 e^{-y^2} dy$, which can be evaluated using parts.

Let

$$u = y \quad dv = 2y e^{-y^2} dx$$

$$du = dy \quad v = -e^{-y^2}$$

$$\text{so } \int_0^{\infty} 2y^2 e^{-y^2} dy = \lim_{k \rightarrow \infty} \left(-ye^{-y^2} \Big|_0^k \right) + \int_0^{\infty} e^{-y^2} dy = 0 + \int_0^{\infty} e^{-y^2} dy.$$

This last integral is well known, with $\int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$, but its evaluation is beyond the first course in calculus. (See <http://courses.ncssm.edu/math/TALKS/PDFS/normal.pdf>)

So, $8.5! = (8.5)(7.5)(6.5)(5.5)(4.5)(3.5)(2.5)(1.5) \frac{\sqrt{\pi}}{2}$, and the “half-factorials” can be computed with the recursive formula

$$f(n) = \begin{cases} \frac{\sqrt{\pi}}{2} & \text{if } n = \frac{1}{2}, \\ n \cdot f(n-1) & \text{if } n > \frac{1}{2} \end{cases}$$

with $n = k + 0.5$ for all non-negative integers k .

Stirling's Formula

When n is large, an efficient way to approximate $n!$ is needed. Stirling's formula is often used. Stirling's formula approximates $n!$ with $\sqrt{2\pi n} e^{-n} n^n$. It is fairly easy to see that a function of the form $e^{-n} n^n$ is appropriate. Consider

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

By taking logs we have

$$\ln(n!) = \ln(n) + \ln(n-1) + \ln(n-2) + \cdots + \ln(3) + \ln(2) + \ln(1).$$

The sum on the right side of the equation can be viewed geometrically as the area of the $n-1$ rectangles under the graph of $y = \ln(x)$. This area can be approximated by the area under the curve (see Figure 1).

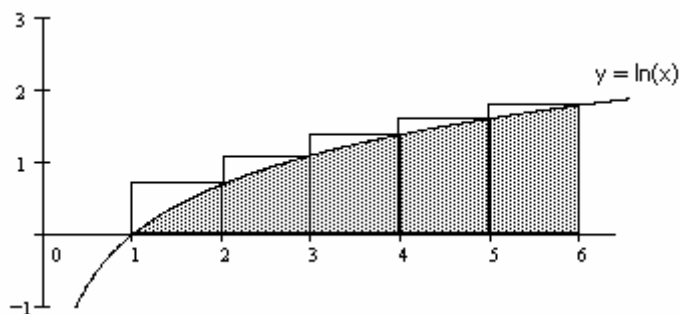


Figure 1

So $\ln(n!) \approx \int_1^n \ln(x) dx$. This integral is another standard integration by parts integral. The value is $\int_1^n \ln(x) dx = n \ln(n) - n + 1$. For large n , the constant 1 is not so important, so $\ln(n!) \approx n \ln(n) - n$. Solving for $n!$, we find

$$(n!) \approx e^{n \ln(n) - n} = e^{-n} e^{\ln(n^n)} = e^{-n} n^n.$$

So something of the form $e^{-n}n^n$ should work. To see where the other terms come from we need a more careful derivation. The following approach is due to mathematician Henry Pollak.

We know that $n! = \int_0^\infty e^{-x}x^n dx$ for all integer values of n . Rewriting, we have $n! = \int_0^\infty e^{-x}e^{n\ln(x)} dx = \int_0^\infty e^{n\ln(x)-x} dx$. Consider the function $f(x) = n\ln(x) - x$. It has its maximum value at $x = n$. Approximate the exponent in the integral with its quadratic approximation centered at its maximum value.

$$\begin{aligned} f(x) &= n\ln(x) - x & f(n) &= n\ln(n) - n \\ f'(x) &= \frac{n}{x} - 1 & f'(n) &= 0 \\ f''(x) &= \frac{-n}{x^2} & f''(n) &= \frac{-1}{n} \end{aligned}$$

So, the quadratic approximation of $f(x) = n\ln(x) - x$ at $x = n$ is

$$n\ln(x) - x \approx n\ln(n) - n - \frac{1}{2} \frac{(x-n)^2}{n}.$$

Therefore,

$$n! = \int_0^\infty e^{n\ln(x)-x} dx \approx \int_0^\infty e^{n\ln(n)-n-\frac{1}{2}\frac{(x-n)^2}{n}} dx = e^{n\ln(n)-n} \int_0^\infty e^{-\frac{1}{2}\frac{(x-n)^2}{n}} dx.$$

We need to evaluate the definite integral and simplify the constant. The integration is a little tricky, requiring two substitutions. Given $n^n e^{-n} \int_0^\infty e^{-\frac{(x-n)^2}{2n}} dx$, we make the substitution $y = x - n$, so we have $n^n e^{-n} \int_{-n}^\infty e^{-\frac{y^2}{2n}} dy$. Now let $u = \frac{y}{\sqrt{n}}$, this gives the integral $n^n e^{-n} \sqrt{n} \int_{-\sqrt{n}}^\infty e^{-\frac{1}{2}u^2} du$. For any reasonably large value of n , this integral has the same value as $n^n e^{-n} \sqrt{n} \int_{-\infty}^\infty e^{-\frac{1}{2}u^2} du$, since the function $g(x) = e^{-\frac{1}{2}x^2}$ dies out so quickly.

We now have the approximation $n! \approx e^{-n}n^n \sqrt{n} \int_{-\infty}^\infty e^{-\frac{1}{2}u^2} du$, which is $n! \approx Ce^{-n}n^n \sqrt{n}$ for some constant C . This is the form of Sterling's formula. This final integral is a variation of the previous standard integral, and has a value of $\int_{-\infty}^\infty e^{-\frac{1}{2}u^2} du = \sqrt{2\pi}$. This gives us Sterling's formula

$$n! \approx e^{-n}n^n \sqrt{2\pi n}.$$

This approximation is good inside the region of the quadratic approximation because that approximation is so good. It is good outside the region of the quadratic approximation because the function values are so small.