Log(Odds) and Quality Control

Quality control techniques must be statistically valid, yet simple enough to implement for a floor worker with limited mathematical background to use. One such method is the Log(Odds) method of quality control. This method involves a combination of conditional probability, odds, and the laws of logarithms to determine a confidence level for the quality of parts which are to be used in the manufacturing process.

Motivating Problem

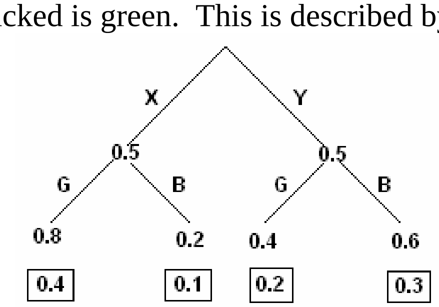
Suppose we have two barrels, **X** and **Y**, filled with both green and blue marbles. Barrel **X** is composed of $80% green marbles and $20% blue marbles, while barrel **Y** is composed of $40% green marbles and $60% blue marbles. Further, suppose that both barrels have a sufficiently large number of marbles in each so that removing one marble does not appreciably affect the probability of getting a marble of a particular color on the next draw. We are going to choose a barrel at random and begin drawing out marbles. We do not know which barrel we have chosen to draw from. If we pull out a green marble, it is more likely that we are drawing from barrel **X** than barrel **Y**, since greens are more predominant in barrel **X** than barrel **Y**. However, if a blue is drawn, barrel **Y** is more likely to be the barrel from which the marbles are being taken. Remember, we do not know from which barrel we are drawing. Another way to look at this is to say that with every green marble drawn, we are more confident that we are drawing from barrel **X** and with every blue marble we are less confident that we are drawing from barrel **X**. As we draw successive marbles, we would like to keep a running chart of the these changes in confidence by keeping track of the probability that we are drawing from barrel **X**, along with the odds of drawing from **X**, and the logarithm of the odds. The reason for the logarithm will be demonstrated shortly. Before any marbles are pulled, we have the situation described in Table 1.

Event	P(**X**)	Odds(**X**)	Log(Odds(**X**))
-	0.5	1	0

Table 1: Initial State

Since we have not yet drawn a marble, it is equally likely that we have picked barrel **X** as barrel **Y**. Thus, the probability of drawing from **X** is 0.5. If $P(X) = 0.5$, then the odds for **X** are 1, and the logarithm of 1 is 0. Here we will use base 10 logarithms, although any base will do.

Now, suppose the first marble picked is green. This is described by the tree diagram in Figure 1.



From this tree diagram, we see that the probability that a marble drawn at random is green and from barrel **X** is 0.4, while the probability that it is green and from barrel **Y** is 0.2. What is the probability that we have drawn from **X** given that we have a green? This is the conditional probability discussed earlier. This probability is given by

$$P(X|G) = \frac{0.4}{0.4+0.2} = \frac{2}{3}$$

The Log(Odds)) table now looks like Table 2.

Event	P(**X**)	Odds(**X**)	Log(Odds(**X**))
-	0.5	1	0
G	0.6667	2	0.30103

Table 2: Draw a Green

The probability that we are drawing from barrel **X** given that we have picked a green marble is $\frac{2}{3}$. The odds for **X** then are 2 to 1. The common logarithm of 2 is 0.30103. The dimension of the Log(Odds(**X**)) column is *bels of confidence*. We say that we now have 0.30103 bels of confidence that we are drawing from barrel **X**.

Now, pick another marble at random. Suppose it, too, is green. We now have

$$P(X|GG) = \frac{0.32}{0.32+0.8} = \frac{4}{5}$$

Figure 2 and Table 3 describe this situation.

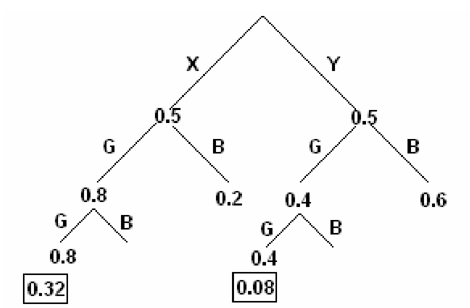


Figure 2: Draw Green then Green

Event	P(**X**)	Odds(**X**)	Log(Odds(**X**))
-	0.5	1	0
G	0.6667	2	0.30103
G	0.8	4	0.60206

Table 3: Draw Green then Green

Again, we have increased the Log(0dds(**X**)) column by .30103 bels of confidence. Now, draw at random again. Suppose it is another green. We have

$$P(X|GGG) = \frac{0.5(0.8)^3}{0.5(0.8)^3 + 0.5(0.4)^3} = \frac{0.256}{0.256 + 0.32} = \frac{8}{9}.$$

The situation is described in Table 4.

Event	P(**X**)	Odds(**X**)	Log(Odds(**X**))
-	0.5	1	0
G	0.6667	2	0.30103
G	0.8	4	0.60206
G	0.8889	8	0.90309

Table 4: Draw Three Greens

As we continue to pull out green marbles, the probability that we have drawn from **X** increases towards one, the odds increase without bound, and the Log(Odds(**X**)) increases with a constant increment of 0.30103.

What happens if we now draw a blue marble? The situation is described in Figure 4. Note that we do not need to draw all of the limbs on our tree model.

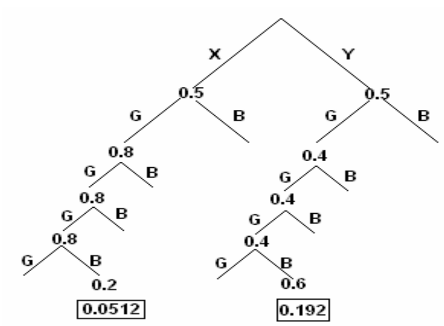


Figure 4: Draw Green, Green, Green, and Blue

The probability of drawing for **X** given that we have GGGB is $\frac{0.0512}{0.0512 + 0.0192} = 0.7272$. The odds are $\frac{8}{3}$ and the Log(Odds(**X**)) is only 0.42597. Before drawing the blue marble, the value was 0.90309. We lost 0.47712 bels of confidence. We are now less sure we are actually picking from barrel **X**.

Suppose that another blue marble is picked.

$$P(X|GGGBB) = \frac{0.01024}{0.010204 + 0.01152} = 0.47058.$$

Since the probability of having drawn from **X** is .47058, the Odds(**X**) = 8/9.

The Log(Odds(**X**)) is -0.051152. Again we have decreased P(**X**), Odds(**X**) and Log(Odds(**X**)). Notice also that the Log(Odds(**X**)) has decreased by the same amount as last time. Evidently, each green marble counts 0.30103 bels for **X** while each blue marble counts 0.47712 bels against **X** (-0.47712).

Every time we pull a green marble out, we have more confidence that we are drawing out of barrel **X** while every time we pull a blue marble out, we have less confidence that the barrel is actually barrel **X**.

Notice that the probability function has a range of [0, 1]. We expand this range to [0, ∞) by using odds. The logarithm function takes [0, ∞) onto the entire real line (-∞, ∞) while giving us constant increments for each green and constant decrements for each blue. Why? And how can this information be used in the plant?

Quality Control

Suppose we are responsible for an automobile manufactoring plant and have placed an order for 12,731,689,000 bolts. We recognize that it is impossible to machine bolts without error, so we expect that some of the bolts are defective. However, we have decided that we will allow only 1% of the bolts to be defective. If the batch that has been shipped to us has fewer than 1% defects, we will accept the order. If, on the other hand, the batch has more than 3% defects, we will not accept the order. Between these two we will negotiate a new price. Of course, we cannot possibly test all the bolts to determine what percentage are actually defective. We can never be certain that the batch has fewer than 1% defects, so we settle for a certain level of confidence that the batch has the appropriate characteristics. Although statistitians frown at the phrase, the laymen would say that he would like to be at least 95% sure that the batch has less than 1% defective bolts. When we use an expression like 95% sure, we mean that the probability that the batch has the appropriate characteristics is 0.95. The statistitian would prefer that the confidence be phrased in probabilistic terminology. However we say it, we can use our Log(Odds) procedure to give us this assurance and allow the workers who use the bolts in the plant to make this important decision. Here's how it works.

Set up two mythical barrels, one (**X**) that is composed of 99% good bolts (think green marbles) and 1% bad bolts (think blue marbles) while the other (**Y**) is composed of 97% good bolts and 3% bad bolts. Have the floor workers begin looking at bolts chosen at random. We would like the probability that we are actually drawing from **X** to be 0.95 (95% sure). As we saw in the marble example, each good bolt will add a certain number of bels of confidence for **X** while each bad bolt will subtract another number of bels of confidence for **X**. Again, we can never know absolutely, but we would like the probability that the batch shipped has the characteristics of barrel **X** rather that barrel **Y** (1% defects rather than 3% defects)to be at least 0.95. As we check bolts, this probability will fluctuate. If, during the process, the probability that we have a shipment which has less than 1% defects ever falls below 0.4, we will discontinue checking bolts and reject the shipment.

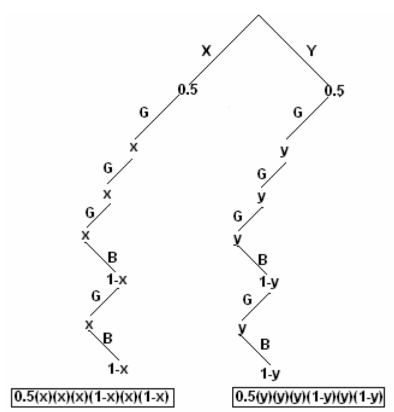
Since we need to have P(**X**) = 0.95, the odds for **X** should be at least 19 to 1. The log of 19 is 1.2787, so we need 1.2787 bels of confidence to accept the shipment.

If P(**X**) = 0.4, then the odds for **X** are 2 to 3. The log of $\frac{2}{3}$ is -0.17609, so if the Log(Odds(**X**)) falls below -0.17609 bels, we reject outright. If, after checking a fixed number of bolts, we have not reached either value, we assume that the batch is somewhere between the two and renegotiate the price.

As the workers check the bolts, each good bolt adds to the Log(Odds(**X**)) score and each bad one subtracts from it. If the running total reaches 1.2787 bels the shipment is accepted. If it falls below -0.17609 it is rejected. If after checking the required number of bolts we have not been convinced that the shipment has the proper characteristic, we renegotiate. If the required confidence criterion was P(**X**) = 0.995, then Odds(**X**) = 199, and we would need 2.29885 bels to accept.

So the workers set about checking bolts. With each good bolt, the bels of confidence increases and with each bad bolt it decreases. But by how much does the level of confidence change? In the first example each blue marble increased the level of confidence by 0.30103 and each green marble would decrease it by 0.47712. Where do these numbers come from? Let's look at a specific example and then generalize.

Suppose that the probability of drawing a good item from bin **X** is *x*. The probability of a bad item then is $1-x$. Also suppose that the probability of drawing a good item from bin **Y** is *y*, while the probability of drawing a bad item is $1-y$. In the marble example, $x=0.7$ and $y=0.4$, while in the bolt example, $x=0.99$ and $y=0.97$. Consider the situation GGGBGBG.



$$P(X|GGGBGBG)=\frac{0.5\left(x\cdot x\cdot x\cdot(1-x)\cdot x\cdot(1-x)\cdot x\right)}{0.5\left(x\cdot x\cdot x\cdot(1-x)\cdot x\cdot(1-x)\cdot x\right)+0.5\left(y\cdot y\cdot y\cdot(1-y)\cdot y\cdot(1-y)\cdot y\right)}$$

Simplifying, we get

$$P(X|GGGBGBG)=\frac{x^5(1-x)^2}{x^5(1-x)^2+y^5(1-y)^2}$$

The odds for **X** are given by

$$\text{Odds}(X) = \frac{x^5(1-x^2)}{y^5(1-y)^2}.$$

Taking the log and simplifying yields

$$\text{Log(Odds}(\mathbf{X})) = \; 5\log x + 2\log(1-x^2) - 5\log y - 2\log(1-y).$$

It seems clear that for each G drawn, the Log(Odds(**X**)) value is increased by $\log x - \log y = \log\frac{x}{y}$ and for each B drawn, by $\log(1-x) - \log(1-y) = \log\frac{1-x}{1-y}$. Notice that $1-y$ is larger than $1-x$ and so $\log\frac{1-x}{1-y} < 0$.

In the marble example, $x = 0.8$ and $y = 0.4$, so each green marble increased the confidence level by $\log\left(\frac{0.8}{0.4}\right) = 0.30103$ and each blue resulted in an increment of $\log\left(\frac{0.2}{0.6}\right) = -0.47712$. In the bolt example, each good bolt would increase the level of confidence by $\log\left(\frac{0.99}{0.97}\right) = 0.00886$ bels of confidence while each bad bolt will count $\log\left(\frac{0.01}{0.03}\right) = -0.47712$ bels.

All that is required of the foreman is to know the acceptance criterion for each shipment and the level of confidence needed to accept or reject. A computer can easily be programmed to add $\log\frac{x}{y}$ and $\log\frac{1-x}{1-y}$ as the bolts are checked. The mathematics is taken out of the floor worker's hands, but the all important inspection is left to the workers themselves.

This paper is based on a talk by Stu Hunter at the Woodrow Wilson Summer Institute in Statistics, July, 1984.

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