Improving Euler with Huen's Method

In Chapter 4, we improved Euler's method by using a quadratic approximation that took into account the concavity of the function. However, we found that the quadratic Euler's method often was very messy and the computation of the derivative is often difficult, particularly if the differential equation is given in terms of both *x* and *y*. Another refinement of Euler's method, known as Huen's method, is twice as accurate as the quadratic Euler's method and requires no derivatives. Huen's method is a clever application of the Fundamental Theorem of Calculus. Recall that the Fundamental Theorem of Calculus states that

$$\text{if } F \text{ is any antiderivative of } f, \text{ then } \int_{x_0}^{x_1} f(x)\,dx = F(x_1) - F(x_0)\,.$$

When solving differential equations, we are generally given two pieces of information, the functional definition of the derivative and an initial point. If we know that $\frac{dy}{dx} = g(x)$ and (x_0, y_0) lies on the curve, then, by the Fundamental Theorem of Calculus,

$$\int_{x_0}^{x_1} g(x)\,dx = y(x_1) - y(x_0)\,.$$

We can rewrite this equation as $y(x_1) = y(x_0) + \int_{x_0}^{x_1} g(x)\,dx$. We know that $y(x_0) = y_0$ and we want to approximate $y(x_1)$ with y_1. We will rewrite the Fundamental Theorem as

$$y_1 = y_0 + \int_{x_0}^{x_1} g(x)\,dx\,.$$

If you are thinking ahead, you will notice that this statement will lead to the iterative equation

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} g(x)\,dx$$

To compute y_1, we need to evaluate the definite integral $\int_{x_0}^{x_1} g(x)\,dx$.

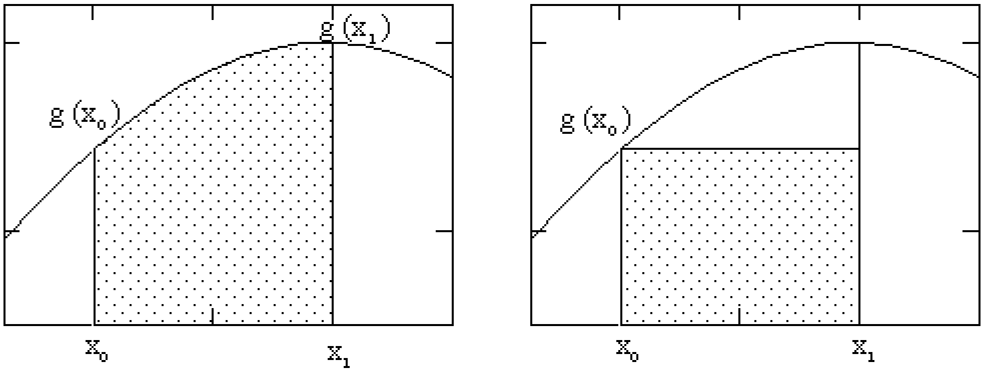

Figure 5.1: The area on the left is approximated by the area on the right

If $\Delta x = x_1 - x_0$ is small, we know that this integral can be approximated the product $g(x_0)\cdot\Delta x$, which gives us the traditional Euler's method iteration

$$y_1 = y_0 + g(x_0)\cdot\Delta x \, .$$

Although Euler's method was originally derived using $\frac{dy}{dx} = g(x)$, we found that it worked equally well with derivatives that we known in terms of both *x* and *y*, that is, $\frac{dy}{dx} = g(x,y)$.

The statement

$$y_1 = y_0 + \int_{x_0}^{x_1} g(x)\,dx$$

means that if the definite integral is evaluated exactly, there is no error in the value y_1. The better the approximation of the definite integral, the better the approximation y_1 is of $f(x_1)$.

We can get a better approximation of the definite intergral by drawing the line passing between the two endpoints (x_0, y_0) and (x_1, y_1) and approximating the value of the integral with the area of the trapezoid created. Figure 5.2: illustrates this idea.

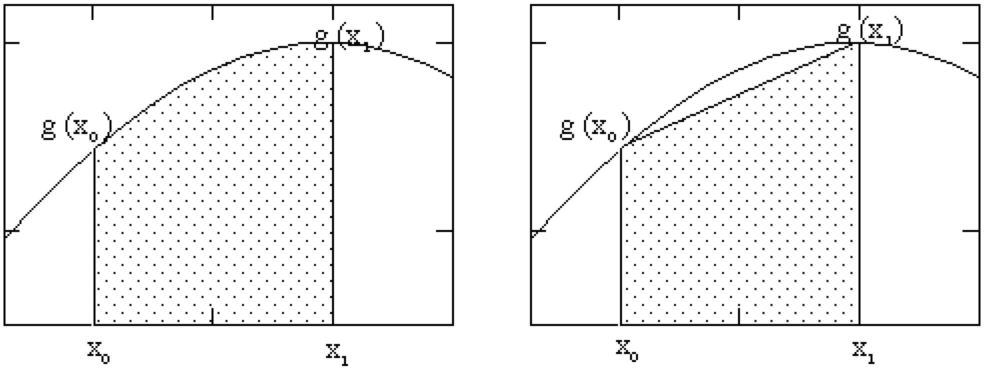

Figure 5.2: The area on the left is approximated by the area on the right

Naturally, this approximation will also be better when the interval is smaller. The area of a trapezoid is given by the length of the base multiplied by the average height of the two sides, in our case this is $A = \Delta x \cdot \frac{1}{2}\left(g(x_0) + g(x_1)\right)$.

So $y_1 = y_0 + \int_{x_0}^{x_1} g(x)\,dx$ can be approximated by

$$y_1 = y_0 + \frac{\Delta x}{2}\left(g(x_0) + g(x_1)\right).$$

(1)

Iterating this scheme produces Huen's method

$$y_i = y_{i-1} + \frac{\Delta x}{2}\left(g(x_{i-1}) + g(x_i)\right). \tag{2}$$

Figure 5.3 compares the error for the solutions to the differential equation $\frac{dy}{dx} = x^2 - 3x$ with $(0,1)$ as the initial condition. The errors shown in the figure are the difference in the curve drawn by Euler's Linear and Quadratic methods and Huen's method and the actual solution function $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1$.

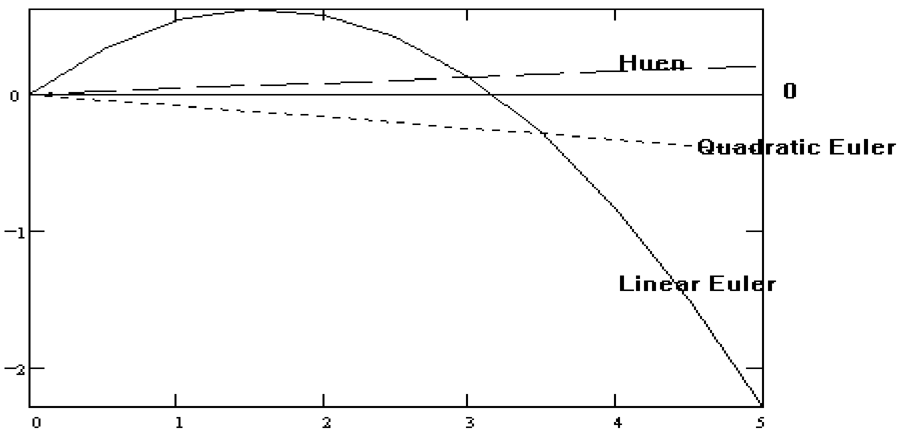

Figure 5.3: Errors for $\frac{dy}{dx} = x^2 - 3x$ with $(0,1)$

As mentioned before, the error from Huen's method is much less than with Euler's method and generally less than half of the error from Euler's quadratic method.

How does Huen's method accomplish such dramatic improvement over Euler without using second derivatives? Look at equation (1) again, this time from a geometric point of view.

$$y_1 = y_0 + \frac{1}{2}\left(g(x_0) + g(x_1)\right)\Delta x . \tag{1}$$

We can think of y_1 as the average of two different approximations:

$$y_1 = y_0 + g(x_0)\Delta x \tag{3}$$

the traditional Euler's approximation, and

$$y_1 = y_0 + g(x_1)\Delta x . \tag{4}$$

Equation (1) can be obtained by adding equations (3) and (4) together and dividing by 2. Each of the equations produces a value for y_1 by starting at y_0 and moving along a tangent line a horizontal distance Δx. They each use a different line. Equation (3) moves along the line tangent to the curve at (x_0, y_0) while equation (4) moves along the line tangent to the curve at (x_1, y_1). How does the tangent at (x_1, y_1) line compare to the tangent at (x_0, y_0)?

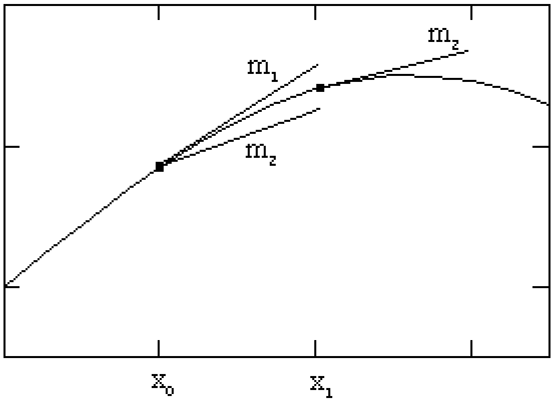

Figure 5.4: At x_0, move off at a slope that is the average of *m1* and *m2*

If the function is concave down (as in Figure 5.4), we know that Euler's method will overshoot the curve. The next slope computed, $m2 = g(x_1)$, will therefore be smaller than the previous slope $m1 = g(x_0)$. By averaging the two slopes, we get a computed value of y_1 that is smaller than that given by Euler's method. If the function is concave up, just the opposite happens, our computed value of y_1 is larger than the value given by Euler's method. It can be shown that Huen's method will always be better than the quadratic Euler's method, and is, in fact, as good as a cubic Euler's method.

How does Huen's method work if the derivative is known in terms of both *x* and *y*? If $\frac{dy}{dx} = g(x,y)$, we simply evaluate *g* at each point (x_i, y_i). This generates the iterative equation

$$y_i = y_{i-1} + \frac{1}{2}\left(g(x_{i-1}, y_{i-1}) + g(x_i, y_i)\right) \qquad (5)$$

If you look carefully at equation (5), you will see a problem. We have defined y_i in terms of itself! How do we get around using y_i on the right hand side of the equation? Huen solved the problem by estimating the value of y_i on the right hand side using Euler's method! That is, Huen defined

$$y_i^* = y_{i-1} + g(x_{i-1}, y_{i-1}) \cdot \Delta x$$

and then defined y_i with

$$y_i = y_{i-1} + \frac{\Delta x}{2}\left(g(x_{i-1}, y_{i-1}) + g(x_i, y_i^*)\right)$$

Writing a single expression for y_i in terms of x_{i-1} and y_{i-1}, we have

$$y_i = y_{i-1} + \frac{1}{2}\left(g(x_{i-1}, y_{i-1}) + g(x_{i-1} + \Delta x, y_{i-1} + g(x_{i-1}, y_{i-1}) \cdot \Delta x)\right) \cdot \Delta x\,.$$

Figure 5.5 illustrates the errors in solving graphically the differential equation $\frac{dy}{dx} = x \cdot y$ with $(0, 1)$ as the initial condition. Again note the marked improvment over Euler's linear method and an error that is approximately half of that produced by Euler's quadratic method.

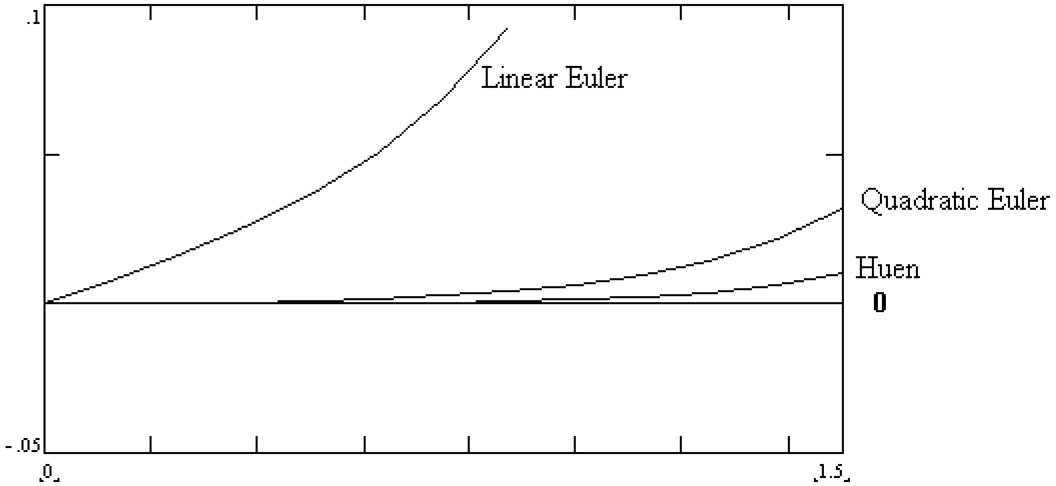

Figure 5.5: Errors for $\frac{dy}{dx} = x \cdot y$ with $(0, 1)$

Reference:

Bartkovich, Kevin, John Goebel, Julie Graves, and Daniel Teague, *Contemporary Calculus through Applications*, Everyday Learning Corporation, Chicago, 1995.