A Local Linearity Approach to Calculus

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The historical role of calculus as the lynch-pin of the mathematical preparation of mathematicians, engineers, and scientists is being pitted against the emerging role of calculus in the general education of the informed citizen. For the calculus instructor, this creates a tension between the need to present arguments that are informative and engaging at the level of the student's understanding, and the desire to present arguments that are mathematically rigorous, but perhaps, not as illuminating. Finding effective ways to introduce the principles of calculus to students from diverse backgrounds and with a multitude of ambitions and employment goals is the challenge many reform calculus texts attempt to meet. The teacher of calculus and the writer of calculus textbooks must decide whether to present arguments that will convince the students they meet in their classes or the colleagues they meet in the hall.

The effect of computer technology on the teaching and learning of calculus has been central to this discussion. Students with a strong calculator-based precalculus experience come to calculus with different skills and conceptions of mathematics. It is important to take advantage of these new skills. This paper will present some of the changes in the instructional development of calculus at the North Carolina School of Science and Mathematics (NCSSM) which utilize the geometric ideas and understandings students have developed when using graphing calculators in preparatory precalculus courses.

Local Linearity

Students who have used graphing calculators extensively in their preparatory courses are familiar with the principle of local linearity. They know from much experience that if they “zoom in” on just about any section of one of the functions they have studied, the function quickly "becomes linear". Calculus adds to this experiential base by explicitly expressing the "zoom line" as the tangent to the curve at $x = a$,

$$y = f'(a)(x-a) + f(a).$$

If the function is indistinguishable from its tangent line in some small region around $x = a$, then the function is said to be locally linear. The principle of local linearity is closely related to that of differentiability; a function is locally linear at all points at which it is differentiable. By emphasizing the visual, geometric aspect of local linearity and the greatly simplified algebraic operations with linear functions, we can offer students intuitively appealing, convincing arguments as a basis for their understanding of calculus. Such an approach falls short of the criterion for rigor desired by many practicing mathematicians, but is in line with Ron Douglas's comment in *Toward a Lean and Lively Calculus* that "I believe that students should learn that the fundamental notion of the differential calculus is how effective linear and quadratic approximation is for studying nice functions and how this can be used to study systems that change."

The mathematics faculty at NCSSM has written a calculus text that, while focusing the use of technology in solving applied problems in calculus, uses the principle of local linearity in it theoretical development of calculus. To illustrate the approach and to open up the discussion, we use as examples three important theorems in elementary calculus; l'Hopital's Rule, the product rule for derivatives, and the Fundamental Theorem of Calculus.

l'Hopital's Rule (Weak Form)

The traditional development follows this general path: **Theorem:** Assume that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

Proof: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)}$ since $f(a) = g(a) = 0$. So

$$\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}=\frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)}.$$

My students often commented after this development, "I see *that* its true, but I don't see *why* its true." What insight does this derivation give the student? Compare the formal proof above to the less formal argument below.

The simplest of all cases is $\lim\limits_{x\to 0}\dfrac{ax}{bx}$. If *f* and *g* are linear, the ratio of *ax* to *bx* is always constant for non-zero *x*, so the limit is just $\dfrac{a}{b}$.

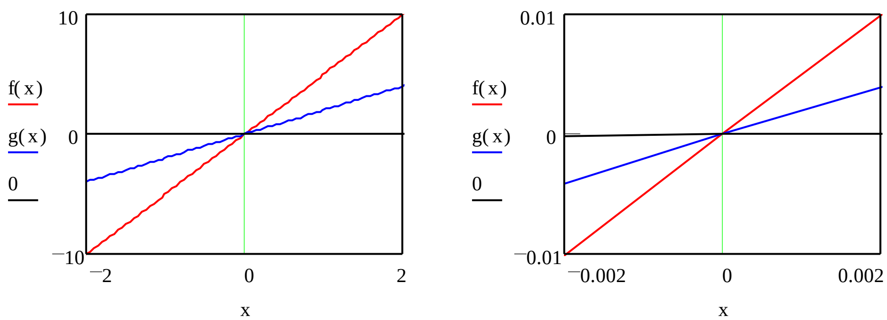

Figure 1: Example of $f(x)=5x$ and $g(x)=2x$ on $[-2,2]$ and $[-0.002,0.002]$

Now, consider $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$. With arbitrary functions *f* and *g*, the ratio of $f(x)$ to $g(x)$ is not constant, but changes with *x*. However, if *f* and *g* are differentiable at $x=0$, they are locally linear near $x=0$ with $f(x)\approx f'(0)x$ and $g(x)\approx g'(0)x$. If you zoom in on *f* and *g* around zero, the geometry gets closer and closer to that of the simple linear case.

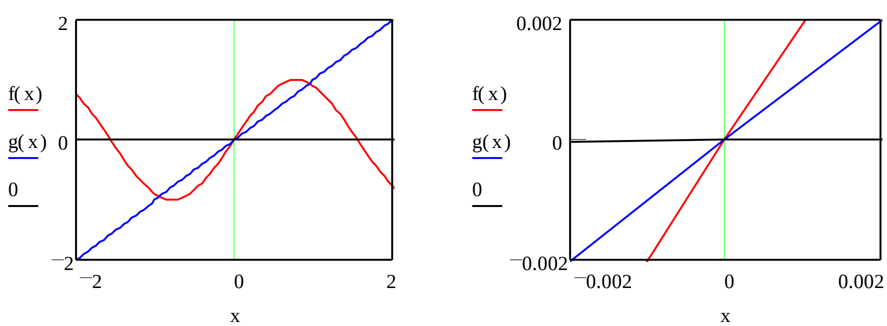

Figure 2: Example with $f(x) = \sin(2x)$ and $g(x) = x$ on $[-2, 2]$ and $[-0.002, 0.002]$

If f and g are well approximated by their tangent lines, we can expect $\frac{f(x)}{g(x)} \approx \frac{f'(0)x}{g'(0)x}$ near $x = 0$. But this is just the simple linear problem, with $\lim_{x\to 0}\frac{f'(0)x}{g'(0)x} = \frac{f'(0)}{g'(0)}$. If we move away from the origin, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

With this argument, students can "see" why l'Hopital's Rule is true. . If you zoom in on differentiable functions, they "behave" locally as if they are linear. As a general approach to a question in calculus, we reduce the problem to its linear approximation, and ask the question of this simple form. In doing this, students can have a tool to assist their investigation into many aspects of differentiable calculus.

There is more to it than this, of course. Henry Pollak once commented that, "In introductory calculus, you cannot tell the whole truth. If you could tell the whole truth, there wouldn't be a course called analysis." Given that you can't tell the whole truth, each of us must decide what part of the truth to tell. Our effort has been to present the part that gives insight and understanding; the part that offers compelling arguments that make the fundamental truths of calculus believable and understandable, and which offer a basis on which to build future work.

The Product Rule

The product rule is often developed in the following way: **Theorem:** Let f and g be differentiable at $x = a$, and let $h(x) = f(x) \cdot g(x)$, then $h'(a) = f'(a)g(a) + f(a)g'(a)$. **Proof:** $h'(a) = \lim_{\Delta x\to 0}\frac{f(a+\Delta x)g(a+\Delta x) - f(a)g(a)}{\Delta x}$. Add and subtract $f(a+\Delta x)g(a)$ in the numerator so that

$$\lim_{\Delta x\to 0}\frac{f(a+\Delta x)g(a+\Delta x)-f(a+\Delta x)g(a)+f(a+\Delta x)g(a)-f(a)g(a)}{\Delta x}.$$

Factoring and rewriting, we have

$$\lim_{\Delta x\to 0}\frac{f(a+\Delta x)\left[g(a+\Delta x)-g(a)\right]}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(a)\left[f(a+\Delta x)-f(a)\right]}{\Delta x}=f(a)g'(a)+g(a)f'(a).$$

The question from students is, "How would you know what to add if you didn't already know the answer?" Some texts describe the addition of 0 as a "useful trick". Having to resort to "tricks" doesn't convince students that mathematics is understandable.

As with l'Hopital's rule, an alternative development begins by considering the linear approximations. If both f and g have derivatives at near $x=a$, then

$$f(x)\approx f'(a)(x-a)+f(a) \text{ and } g(x)\approx g'(a)(x-a)+g(a).$$

The function $h(x)$, being the product of $f(x)$ and $g(x)$, can be approximated near $x=a$ by the product of the two linear approximations, so that

$$h(x)\approx\left[f'(a)(x-a)+f(a)\right]\cdot\left[g'(a)(x-a)+g(a)\right].$$

That there are better quadratic approximations to h is a matter to be taken up later.

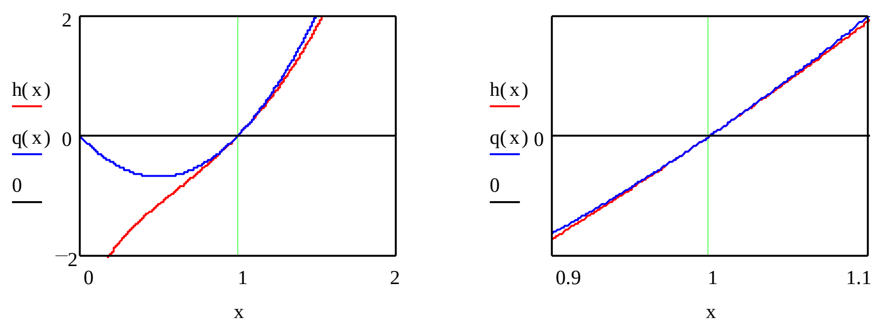

Figure 3: $h(x)=\ln(x)\cdot e^{x}$ at $x=1$ and the quadratic approximation $q(x)=(x-1)\cdot(e\,x)$

This product of the linear approximations simplifies to

$$h(x)\approx f(a)g(a)(x-a)^2+f(a)g'(a)(x-a)+f'(a)g(a)(x-a)+f(a)g(a),$$

so

$$h'(x)\approx 2f(a)g(a)(x-a)+f(a)g'(a)+f'(a)g(a)$$

for values of x near a. However, at $x=a$, the approximations are exact, and

$$h'(a) = 2f(a)g(a)(a-a) + f(a)g'(a) + f'(a)g(a),$$

which gives the product rule

$$h'(a) = f(a)g'(a) + f'(a)g(a) \tag{1}$$

Our original choice of $x = a$ was arbitrary and depended only on both functions f and g having derivatives at $x = a$; therefore, the relationship in equation (1) holds for any x where the functions f and g differentiable.

By using the linearity approach, students can derive the product rule themselves, without knowing the result ahead of time. Students learn to use linear approximations as a guide to understanding the behavior of nonlinear functions. In so doing, they have learned more than a result, they have developed a general problem-solving technique through which they can investigate the subject of calculus.

The Fundamental Theorem of Calculus

Rather than approach the fundamental theorem of calculus through an area argument, we utilize the student's understanding of Euler's method as a path to this important theorem. Euler's method is yet another application of linear approximations.

Given a differential equation $\frac{dy}{dx} = f'(x)$ and an initial condition $y_0 = f(x_0)$, Euler's method allows us to generate a sequence of values, y_n, that approximate the values of $f(x_n)$ by iterating the equation $y_n = y_{n-1} + f'(x_{n-1}) \cdot \Delta x$. The accompanying values of x_n are produced by iterating the equation $x_n = x_{n-1} + \Delta x$. The sequence of values,

$$\begin{gathered} y_1 = y_0 + f'(x_0) \cdot \Delta x, \\ y_2 = y_1 + f'(x_1) \cdot \Delta x, \\ y_3 = y_2 + f'(x_2) \cdot \Delta x \\ \vdots \\ y_n = y_{n-1} + f'(x_{n-1}) \cdot \Delta x \end{gathered}$$

allows us to find approximate solutions to a number of challenging differential equations. However, we can rewrite these expressions to achieve a different purpose. Substituting the expression for y_1 into the equation for y_2 yields

$$y_2 = y_0 + f'(x_0) \cdot \Delta x + f'(x_1) \cdot \Delta x.$$

Continuing in this manner, after n iterations an approximation y_n for $f(x_n)$ is given by

$$y_n = y_0 + \sum_{i=0}^{n-1} f'(x_i)\cdot\Delta x .$$

Subtracting y_0 from both sides gives $y_n - y_0 = \sum_{i=0}^{n-1} f'(x_i)\cdot\Delta x$.

The difference $y_n - y_0$ is an approximation for the net change in $f(x)$ from x_0 to x_n. If we know a formula for $f(x)$, then the actual net change in height is $f(x_n) - f(x_0)$. So

$$f(x_n) - f(x_0) \approx \sum_{i=0}^{n-1} f'(x_i)\cdot\Delta x .$$

Further, we know that $y_0 = f(x_0)$ while $y_n \approx f(x_n)$.

The Euler approximations improve as Δx gets smaller. As Δx goes to zero, y_n approaches $f(x_n)$, so that $\sum_{i=0}^{n-1} f'(x_i)\cdot\Delta x$ gets closer to the actual value of $f(x_n) - f(x_0)$. Indeed,

$$\lim_{\Delta x\to 0}\sum_{i=0}^{n-1} f'(x_i)\cdot\Delta x = f(x_n) - f(x_0), \qquad (2)$$

where $\Delta x = \dfrac{x_n - x_0}{n}$ and $x_i = x_{i-1} + \Delta x$.

We then define the definite integral $\int_a^b f'(x)dx = \lim_{\Delta x\to 0}\sum_{i=0}^{n-1} f'(x_i)\cdot\Delta x$, where a is x_0 and b is x_n. Since the right side of equation (2) is the net change in $f(x)$ from $x = a$ to $x = b$, which is $f(b) - f(a)$, the definite integral on the left side of this equation also equals the net change in $f(x)$, so that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Conclusion

The linearity approach takes advantage of and builds upon the geometric intuition developed by using graphing tools in preparatory courses. Entering students are comfortable with the both the principle of local linearity and the algebraic manipulations of linear functions. By considering the behavior of the tangent lines, students can go a long way towards determining the behavior of nonlinear functions and the rules that govern differential calculus. If greater rigor is desired, this approach is a very useful way to investigate the problem. Students can reduce the problem to the linear approximation, "follow the lines", and make a conjecture about how well the behavior of the linear model represents the behavior of the function. They now have a conjecture which requires more rigorous justification.

Each of the calculus reform projects is experimenting with both content and pedogogy. Our students' experience with this approach has been overwhelmingly positive. It is particularly positive for those students who have previously had an introductory calculus experience that was too rigorous for the students level of mathematical sophistication. By sharing our ideas, learning from each other, and combining ideas, perhaps we will create a calculus course that satisfies the competing factions in the calculus debate.

References

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