Graphical Representations of Systems of Linear Equations

appeared as

Teague, Daniel J., "Systems of Linear Equations", *Centroid*, NCCTM, Volume XVIII, Number 2, Fall 1991.

One of the great advantages of teaching with the computer is the quick and easy access to the graphical representations of algebraic equations. In many situations, the graph offers much more insight into the problem than does the algebra. This module concerns a group of problems for which the graphs offer not only the insight into the solution, but pose the problems as well.

We will first consider a system of linear equations in general form, $ax + by = c$, where *a*, *b*, and *c* are in arithmetic progression. One example is $2x + 4y = 6$. Another is $3x - y = -5$. If we graph these two equations (Figure 1), it is not surprising that they intersect, since the slopes of the lines are different.

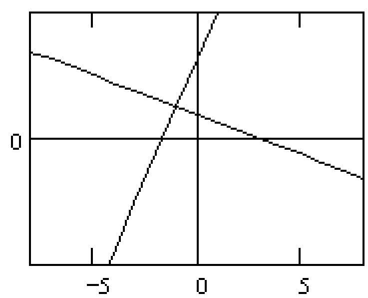

Figure 1

If we now add to the system several more equations, as indicated in Figure 2, we quickly see that something special, possibly surprising, is happening; something that is not obvious from looking at the equations.

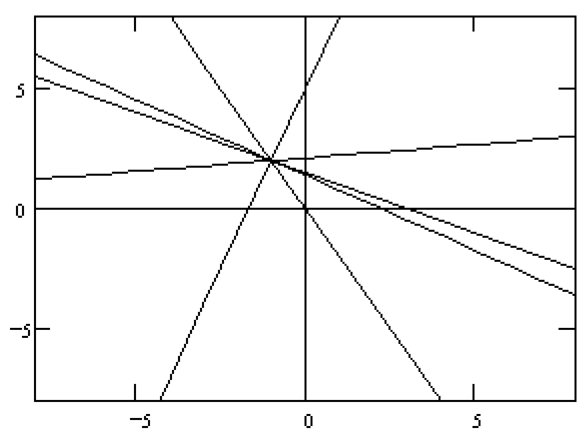

Figure 2

It is a simple task to solve the system to determine that the point $(-1, 2)$ lies on all linear equations in general form whose coefficients form an arithmetic progression. Why should this be so? If you consider the form of these equations, $ax + (a+k)y = a + 2k$, the answer is clear. The right side of the equation has 2*k*, while the only term with *k* on the left side is associated with y. For the two sides to be equal, *y* must equal 2. But if $y = 2$, then on the left side we have $ax + (a+k)2 = a + 2k$, and *x* must equal -1. The important aspect of this problem is that the geometry of the system, not the algebra, was the interesting and surprising part.

The next question to consider is, what happens if you try other patterns for *a*, *b*, and *c*? Suppose we consider the system of linear equations in general form $ax + by = c$, where *a*, *b*, and *c* form a geometric progression. What geometry does this system have? If you begin by graphing equations of this form, there seem to be intersections all over the plane.

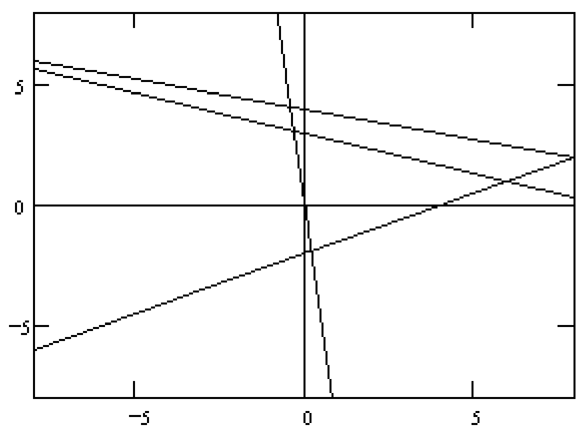

Figure 3

However, as more equations are graphed, something in the graph begins to take shape.

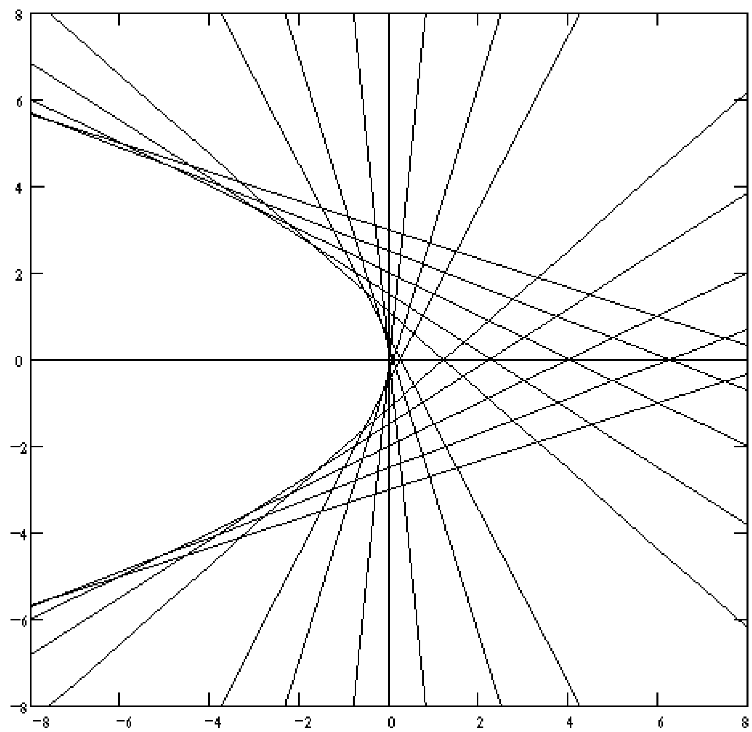

Figure 4

Notice that there is a region in the plane which has no intersections! What region is this, and from where does it come? Solving the general system $ax + ary = ar^2$ and $bx + bsy = bs^2$ gives solutions of $x = -rs$ and $y = r + s$. How do these values of *x* and *y* determine the "null" region in the plane? Students can play with the graph and guess the boundary for the region. It appears to be a parabola of the form $y^2 = -kx$, for some positive number *k*. Trying several values of *k*, students will quickly see that $k = 4$ gives a good fit.

This boundary curve can be determined analytically by considering the equation $ax + ary = ar^2$ as a quadratic function of *r*. Then $r^2 - yr - x = 0$. Solving for *r*, we find that $r = \dfrac{y \pm \sqrt{y^2 - 4x}}{2}$. Notice that no *r* exists for which $y^2 - 4x < 0$. This means that no linear equation of the form $ax + ary = ar^2$ passes through the region defined by $y^2 < -4x$. Notice that the two solutions $x = -rs$ and $y = r + s$ satisfy this condition. That is, $(r + s)^2 \geq -4(-rs)$. To see this, combine all terms so that $r^2 + 2rs + s^2 - 4rs \geq 0$, and notice that the left side of the equation is a perfect square $r^2 - 2rs + s^2 = (r - s)^2$ which is always greater than or equal to zero. The boundary for the "null" region seen in Figure 4 is $y^2 = -4x$. Again, there is nothing in the algebraic representation of this system to indicate something quite spectacular is happening. Both the problem and its solutions are motivated by the graphics.

Other similar questions naturally follow. Determine and explain the solution patterns defined by considering the system $ax + by = c$, where *a*, *b*, and *c* are defined by:

1. $a \cdot b = c$, for differing values of *c*. eg. $2x + 5y = 10$, $3x - 4y = -12$, and $6x + y = 6$

2. fix *c*, ($c = 48$, for example) and $a \cdot b = c$. eg $4x + 12y = 48$, $-3x - 16y = 48$, and $12x + 4y = 48$

3. $a^2 + b^2 = c^2$

4. $a = \sqrt{b}$ and $b = \sqrt{c}$

5. What figures are created if *x* is replaced by x^2 in these equations?

Reference: Mulligan, Catherine, and Anita Wah, *String Art Design: A Diary of Computer Discovery*, in **Algebra Teachers Are the Keys**, the Woodrow Wilson National Fellowship Foundation Mathematics Institute 1989 Curriculum Module, Princeton, New Jersey.

Dan Teague

The North Carolina School of Science and Mathematics

P.O. Box 2418
1219 Broad Street
Durham, North Carolina 27715