Playing With Pixels: A Computer Investigation

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Introduction

One of the primary goals of mathematics reform is that students see mathematical thinking as a process rather than a product. The mathematics lies in the solving, not in the solution. Equally important, students should realize that the process of employing mathematics to solve a problem is not generally a straight-forward one. Students should not question their abilities in mathematics because they have to work at solving problems; we all work at solving problems. An essential ingredient in mathematical investigations often missing in my classroom is the critical role that errors and missteps play in searching for answers to new questions. Students often think that the goal of mathematics is to get the right answer without making any mistakes. Mistakes and missteps are not the enemy in a mathematical investigation. Henry Ford could easily have been talking about mathematical investigations when he said, "Failure is only the opportunity to begin again. ... What is past is useful only as it suggests ways and means for progress." This article is a history of a mathematical investigation, as it happened, complete with missteps, temporary setbacks, and misunderstandings.

The Origin of the Problem

Many of the modern graphing software packages allow the user to specify the number of points used in construction the graph. The Mathematics Exploration Toolkit (MET) uses the command **gra** *n* to do this. Earlier, I had read an article by Frank Demanna and Bert Waits in *The College Mathematics Journal* describing the difficulty in graphing accurately periodic functions with a small period relative to the size of the viewing window. Generally, you just get a mess if you try to create such graphs. I was attempting to determine the number of points necessary to get a "good" graph for such functions as $f(x) = \sin(75x)$. As part of that investigation, I happened to graph $y = \sin(75x)$ on the interval $[-2p, 2p]$ using the command **gra** 151. What appeared on the screen is shown in Figure 1.

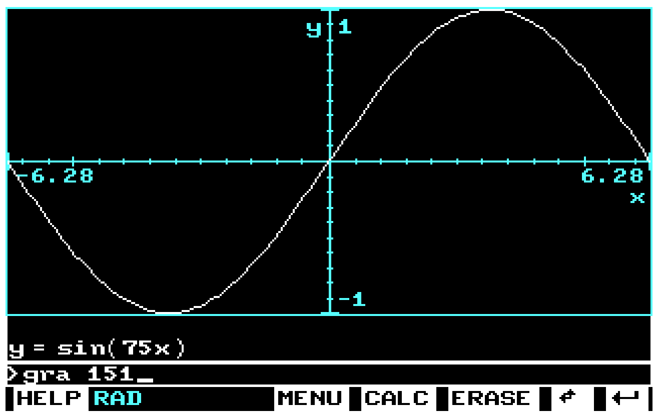

Figure 1: $y = \sin(75x)$ on $[-2p, 2p]$ using the command **gra** 151

First of all, this wasn't the usual erratic jumble of line segments that I was used to seeing. Secondly, although the graph is a nice, smooth sine function, it is not the graph of $y = \sin(75x)$. Instead, it appears to be the graph of $y = \sin\left(\frac{x}{2}\right)$. Why was $y = \sin\left(\frac{x}{2}\right)$ graphed instead of $y = \sin(75x)$? This was the question I wanted to investigate.

Further Exploration and The First Conjecture

Fiddling some more, I graphed $y = \sin(20x)$ over the domain of $[-3p, 3p]$. To produce a single sine wave over the given domain required the command **gra** 61, or 61 points in the graph. The function $y = \sin(41x)$ over the domain of $[-2p, 2p]$ required 83 points.

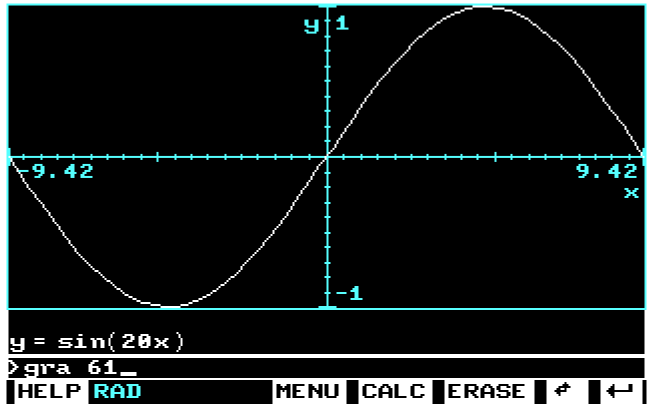

Figure 2: $y = \sin(20x)$ on $[-3p, 3p]$ using the command **gra** 61

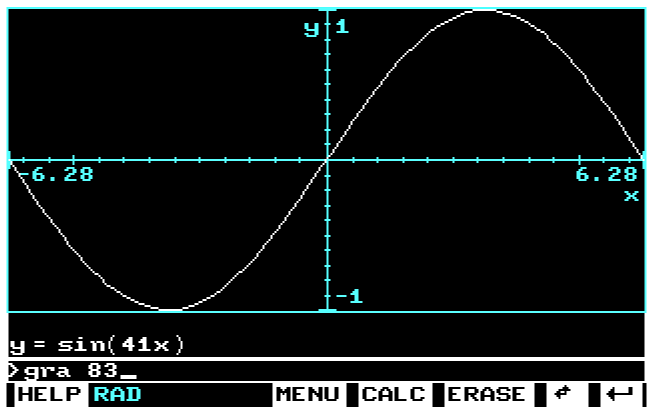

Figure 3: $y = \sin(41x)$ on $[-2p, 2p]$ using the command **gra** 83

Several other graphs with differing values of k and d sufficed to convince me that to achieve a single singe wave in the viewing window $[-dp, dp]$, simply graph $y = \sin(k\,x)$ using the MET command **gra** $(kd + 1)$.

The First Error

I wanted to show what I had found to a colleague, so I asked her to give me a function $y = \sin(k\,x)$ and an interval $[-dp, dp]$ and I'd graph the function so it would appear as a single sine wave across the viewing window, that is, it would look like I had graphed $y = \sin\left(\frac{x}{d}\right)$. Fortunately, she chose the function $y = \sin(21x)$ on the interval $[-p, p]$. I proudly used **gra** 22 to produce the graph in Figure 4.

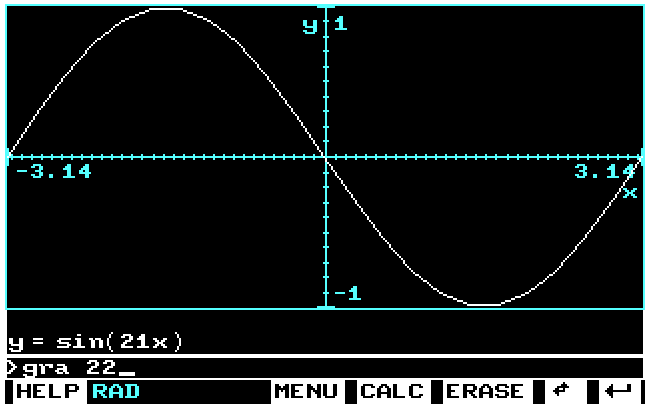

Figure 4: $y = \sin(21x)$ on $[-p, p]$ using the command **gra** 22

Further Investigation

Clearly, more was happening than I had first realized. Why had it worked on all of the functions and intervals I had tried before, but didn't work on this one? What was different about $y = \sin(21x)$? By looking over the list of functions and intervals I had used originally, I found that, without intending to, all of the choices I had originally made had the common feature that the product *kd* was even. To achieve the desired result with $y = \sin(21x)$ on the interval $[-p, p]$, you must specify **gra** 20. The function $y = \sin(75x)$ over the domain of $[-p, p]$ required **gra** 74, while the graph of $y = \sin(41x)$ over the domain of $[-3p, 3p]$ required **gra** 122.

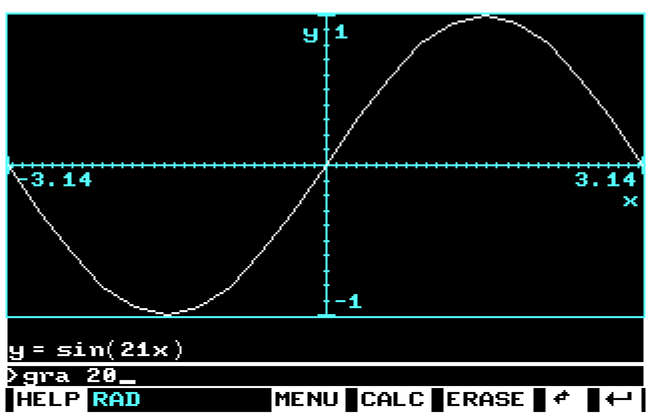

Figure 5: $y = \sin(21x)$ on $[-p, p]$ using the command gra 20

Figure 6: $y = \sin(75x)$ on $[-\pi, \pi]$ using the command gra 74

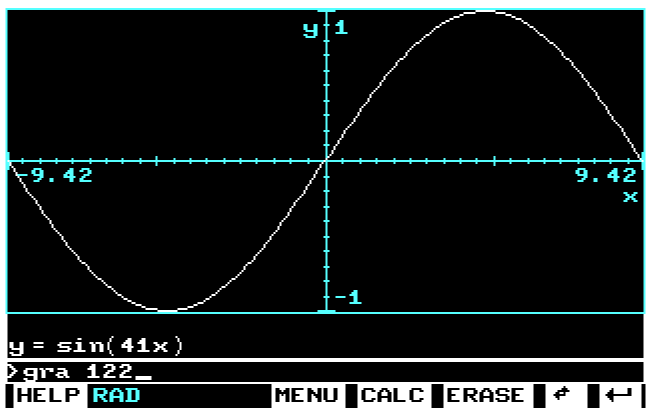

Figure 7: $y = \sin(41x)$ on $[-3\pi, 3\pi]$ using the command gra 122

The Second Conjecture

After about two dozen graphs, I believed that the true pattern was beginning to emerge. My experimental, empirical investigation resulted in a conjecture:

> The graph of $y = \sin(k\,x)$ graphed over the domain $[-d\pi, d\pi]$ with n points (**gra** *n*) would appear indistinguishable in the viewing window from the graph of $y = \sin\left(\frac{x}{d}\right)$ whenever $n = kd + 1$ if kd is even and $n = kd - 1$ if kd is odd.

Why would this be true? It would be true if the equation $\sin(k\,x) = \sin\left(\frac{x}{d}\right)$ has $kd \pm 1$ equally spaced solutions in the given interval, and the grapher evaluates the function $y = \sin(k\,x)$ only at these points. As a result, the graph on the screen would appear to be the graph of $y = \sin\left(\frac{x}{d}\right)$.

Analysis: Turn Off the Computer and Bring Out the Pencil and Paper

My hypothesis is that the equation

$$\sin(kx) - \sin\left(\frac{x}{d}\right) = 0 \tag{1}$$

has $kd + 1$ solutions in the interval $[-dp, dp]$ if kd is even or $kd - 1$ solutions in the interval $[-dp, dp]$ if kd is odd. As with many other situations in algebra, we prefer to determine the values that make a product equal to zero rather than a sum. For algebraic functions, we accomplish this by factoring. With trigonometric equations, we use trigonometric identities. Solving equation (1) requires the trigonometric identity

$$\sin(Ax) - \sin(Bx) = 2 \cdot \cos\left(\frac{A+B}{2}x\right) \cdot \sin\left(\frac{A-B}{2}x\right). \tag{2}$$

Substituting $A = k$ and $B = \frac{1}{d}$ into equation (2), we have

$$2 \cdot \cos\left(\frac{kd+1}{2d}x\right) \cdot \sin\left(\frac{kd-1}{2d}x\right) = 0. \tag{3}$$

Setting $\sin\left(\frac{kd-1}{2d}x\right) = 0$, we know that either $x = 0$ or $\frac{kd-1}{2d}x = ap$ for $a = \pm 1, \pm 2, \pm 3, \cdots$. Solving for *x* we find that $x = \frac{2p\,ad}{kd-1}$. We want to know the number of solutions in the interval $[-dp, dp]$, that is, we want to find the largest value of *a* so that $x \le dp$. Solving this inequality gives $a \le \frac{kd-1}{2}$. Since there are as many solution for $x < 0$ as for $x > 0$, we have $2a+1$ solutions on the interval altogether (counting $x = 0$). There are $2a+1 = 2\left(\frac{kd-1}{2}\right) = kd$ solutions to $\sin\left(\frac{kd-1}{2d}x\right) = 0$, where kd is odd.

The Second Error and Back to the Computer Graphs

Unfortunately, I was expecting either $kd-1$ or $kd+1$ solutions. Where did I go wrong this time? After carefully reviewing the algebra and convincing myself that I had counted the solutions correctly, I returned to the graph. Maybe I didn't really understand what was happening with the computer. More fiddling eventually led to the graph of $y = \sin(7x)$ on the interval $[-p, p]$ using the command **gra** 6. This graph finally gave the answer I needed.

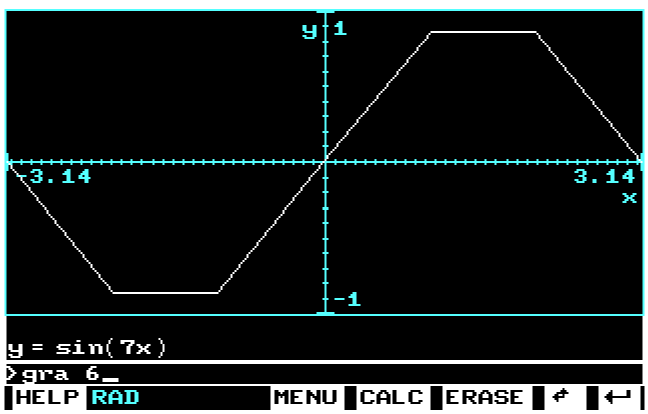

Figure 6: $y = \sin(7x)$ on $[-p, p]$ using the command **gra** 6

Understanding and the Third Conjecture

The graph clearly shows that the **gra** 6 command uses 6 **intervals** in its graph. I had misunderstood the command **gra** *n*, believing the software used *n* **points**. On MET, the command **gra** *n* specifies *n* intervals and therefore, $n+1$ points. I wasn't graphing $y = \sin(7x)$ with 6 points at all, but with 7 points. In my earlier graphs, I had not been using $kd-1$ and $kd+1$ points, but kd and $kd+2$ points! My most recent conjecture was also incorrect. What I had just shown is that:

> The graph of $y=\sin(k\,x)$ appears indistinguishable from the graph of $y=\sin\left(\frac{x}{d}\right)$ on the domain $\left[-dp\,,dp\right]$ with kd odd, if the graphing tool uses kd equally spaced points to construct the graph.

The grapher evaluates the function $y=\sin(k\,x)$ only at the kd points where $\sin(k\,x)=\sin\left(\frac{x}{d}\right)$ and, as a result, the graph on the screen appears to be the graph of $y=\sin\left(\frac{x}{d}\right)$.

What about the graphs with kd even? Considering the solutions to $\cos\left(\frac{kd+1}{2d}x\right)=0$ in the interval $\left[-dp,dp\right]$, we find that $\frac{kd+1}{2d}x=\left(2a-1\right)\frac{p}{2}$. This implies that $x=\frac{dp\left(2a-1\right)}{kd+1}$. Solving for $a\le dp$, we find that $a\le\frac{kd+2}{2}$. Again, there are just as many solutions for $x<0$ as for $x>0$, so there are $2a$ or $kd+2$ solutions in the desired interval. When we graph $y=\sin(k\,x)$ using $kd+2$ points over the interval $\left[-dp,dp\right]$ and kd is even, the grapher only plots those points where $\sin(k\,x)=\sin\left(\frac{x}{d}\right)$, and therefore, the graphs appears to be the graph of $y=\sin\left(\frac{x}{d}\right)$.

Testing the Conjecture with Calculators

The TI-81 has 96 columns on its graphing window. In this case, the number of points to be graphed is fixed and even. Our analysis suggests that we can achieve the same effect on the TI-81 by graphing $y=\sin(94x)$ on the interval $\left[-p,p\right]$, $y=\sin(47x)$ on the interval $\left[-2p,2p\right]$, and $y=\sin(31.333x)$ on the interval $\left[-3p,3p\right]$. In each case, the graph appears to be the graph of $y = \sin\left(\frac{x}{d}\right)$ over the interval $\left[-dp, dp\right]$. In each case, $kd = 94$ and $n = 96$.

The Casio *fx* 7700 calculator has 94 columns on its graphing screen. The graph of $y = \sin(92x)$ on $\left[-p, p\right]$ should appear to be the graph of $y = \sin(x)$, and the graph of $y = \sin(47x)$ on $\left[-2p, 2p\right]$ should appear to be the graph of $y = \sin\left(\frac{x}{2}\right)$.

Theorem

After many missteps, I believe I know what is happening:

> The graph of $y = \sin(k\,x)$ appears indistinguishable from the graph of $y = \sin\left(\frac{x}{d}\right)$ on the domain $\left[-dp, dp\right]$ if the graphing tool uses kd evenly spaced points when kd is odd and $kd + 2$ evenly spaced points when kd is even.

Some questions left unanswered

1) On the TI-82, the graph of $y = \sin(95x)$ on the interval $\left[-p, p\right]$ appears in the viewing window to be the graph of $y = \sin(x)$. How many columns of pixels does the TI-82 use in graphing?

2) Just how many solutions are there to the equation $\sin(k\,x) - \sin\left(\frac{x}{d}\right) = 0$ on the interval $\left[-dp, dp\right]$? We never used the solutions to $\cos\left(\frac{kd+1}{2d}x\right) = 0$ when kd was odd or the solutions to $\sin\left(\frac{kd-1}{2d}x\right) = 0$ when kd was even. What happens to them?

3) Since the solutions to $\cos\left(\frac{kd+1}{2d}x\right)=0$ lie between the solutions to $\sin\left(\frac{kd-1}{2d}x\right)=0$, what phase shift would produce the same graph?

4) What is the relationship between *k*, *d*, and *n*, so that the graph of $y=\sin(kx)$ graphed using n point on the interval $[-dp, dp]$ is indistinguishable from $y=\sin(x)$?

5) Do similar graphing anomalies exist with the other trigonometric functions?

Reference:

Demanna, Frank and Bert Waits, "Pitfalls in Graphical Computation", *The College Mathematics Journal*, Volume 19, Number 2, March, 1988.