Finding Zeros Using Newton's Method

Do you remember in Algebra II when you used the quadratic formula to find zeros of quadratic functions? It was a guaranteed method of finding zeros of those functions whether they factored nicely or not. Perhaps you also learned the Rational Roots Theorem, Descartes’ Law of Signs, and other techniques to help you find zeros of polynomials of higher degree. These methods work nicely for find the zeros of polynomial functions only, but we often need to find zeros of functions that are not polynomials. Why are we interested in these values (also known as roots)?

Here are some very important reasons:

- The zeros of a function $f(x)$ tell us where the graph of the function might change from being positive to negative or vice versa. This tells us when the graph might cross the $x$-axis.

- The zeros of the derivative of a function, $f'(x)$, tell us where the graph of $f(x)$ might change direction (increasing to decreasing or vice versa), which helps us determine the location of maxima and minima of the function $f(x)$. This skill is helpful when working with applications in optimization.

- The zeros of the second derivative of a function, $f''(x)$, tell us where the graph of $f(x)$ might change concavity. This helps us determine where the graph might have inflection points. This is really helpful when we are interested where the rate of change of a function is changing the fastest, such as when the acceleration of a vehicle is the highest.

There are a number of methods used to find zeros of functions:

1. Use computer technology or graphing calculators to approximate zeros using calculation tools. It should be noted that this technology employs algorithms similar to ones described below to approximate zeros.

2. Use the **bisection algorithm** for continuous functions. The algorithm proceeds as follows:

   **Step 1:** Since we know that a zero of a continuous function occurs when that function changes from positive to negative or vice versa, we begin by choosing any two $x$-values whose corresponding $y$-values have different signs.

   **Step 2:** Find the midpoint $x$-value between the two $x$-values from Step 1, and calculate its $y$-value.

   **Step 3:** Replace the original $x$-value from Step 1 that has a $y$-value of the same sign as the midpoint value from Step 2.

   **Step 4:** You now have two $x$-values – one from Step 1 and the one found in Step 2. These two $x$-values should have function values of opposite sign. Repeat the "midpoint" process with these two $x$-values.

   **Step 5:** Proceed in the same way by creating a new interval between $x$-values that have function values of opposite sign that is half as long as the previous interval until the length of the interval gets small enough to identify the zero with the desired accuracy.

   *Adapted from Contemporary Calculus through Applications, NCSSM Mathematics Department, 1996*

**Bisection Algorithm Applet:** The following link is an interactive applet created by Joseph L. Zachary that allows you to visualize the bisection method:

3. Use the **secant algorithm**, which employs the use of secant lines to approximate the value of a function at certain points. This may seem familiar since a similar method can be used in single-variable calculus to approximate the slope of tangent lines before the concept of derivative has been formally developed. This algorithm proceeds as follows:

**Step 1:** Choose any two points on the graph of a function and find the equation of the secant line connecting those two points. Find the x-intercept (or zero) of this line and call it $x_1$.

**Step 2:** Calculate $f(x_1)$, and replace one of the points from Step 1 with the new point $(x_1, f(x_1))$. Find the secant line connecting this point and the one left over from Step 1, and calculate the x-intercept of this line.

**Step 3:** Replace the point replacement process described in Step 2 with this new x-intercept and one of the points from Step 2.

**Step 4:** Repeat this process until you find an x-intercept with the desired accuracy.

This method hinges on the idea that the x-intercept of the final secant line will be very close to the zero of the original function. A benefit of this method over the bisection method is that you don't have to choose x-values that have function values of opposite signs!

4. Like Methods 2 and 3 (bisection and secant algorithms), **Newton’s Method** is iterative, but it requires some calculus skills. It is closely related to the secant algorithm but uses tangent lines rather than secant lines to approximate zeros.

Newton’s method is nicely demonstrated by the following example. **Example:** Use Newton's Method to find a zero of the function $f(x) = x^2 - 7$.

**Solution:** Since we know that the zeros of this function are $±\sqrt{7} ≈ ±2.6458$, we will be able to compare the zero generated by Newton’s Method to these values.

**Step 1:** Draw the function on your paper, and make an initial guess of one of the zeros. Let’s choose $x = 1$.

**Step 2:** Find the equation of the tangent line to this function at the chosen point, and draw it on your graph. Recall from our study of derivatives that the slope of the tangent line is found by calculating the derivative of the function at the point of interest. So, since $f'(x) = 2x$, we have that $f'(1) = 2$. Since $f(1) = -6$ in this example, we can verify the tangent line written above. The tangent line to the function at this point is $y = 2(x - 1) - 6$.

**Step 3:** Find the x-intercept of the tangent line from Step 2. For our line $y_1$, we find that the x-intercept is $x = 4$. (Verify this!)

**Step 4:** Replace the initial guess of the zero with this x-intercept, and repeat the process of finding the x-intercept of the tangent line to the function at the new point. Draw this new line on your graph. The equation of the tangent line to $f(x)$ at $x = 4$ is $y_2 = 8(x - 4) + 9$. (Verify this, too!) The x-intercept of this line is 2.875. You should notice that this x-intercept is closer to the actual zero on the function than your initial guess and the x-intercept from Step 3!

**Steps 5 and on:** Continue this process until you obtain a zero of the desired accuracy. If we continue this process through 5 iterations, we will find that the last two x-intercepts obtained are the same up to their thousandths place, $x = 2.646$. We find that $f(2.646) ≈ 0.001$, which is very close to 0!
Generalization of Newton's Method

We can actually generalize this result! Recall that we made an initial guess of a zero of the function of interest and generated the tangent line to the function at that guessed point. So, we began with the following line:

\[ y_1 = f'(x_0)(x-x_0) + f(x_0) \]

We found the \(x\)-intercept of this line by setting the line equal to zero:

\[ 0 = f'(x_0)(x-x_0) + f(x_0) \]
\[ f'(x_0)(x-x_0) = -f(x_0) \]
\[ x = x_0 - \frac{f(x_0)}{f'(x_0)} \]

Suppose we let this \(x\) be known as \(x_1\). So, \(x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}\). We then found the tangent line to the original function at this new point \(x_1\), which is \(y_2 = f'(x_1)(x_2-x_1) + f(x_1)\).

Solving for the \(x\)-intercept of this function, we find that \(x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}\). Proceeding in this way, we can generalize the sequence of approximations of the zero of the original function by the following!

\[ x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \]

For our purposes, we will perform as many iterations of this method that provides us with 3 decimal place accuracy. In other words, if two successive iterations of the method produce the same approximation up to their thousandths place, we can stop.

Newton's Method Applets

Click on the following links to access interactive applets that allows you to visualize how Newton's Method works.

http://math.furman.edu/~dcs/java/newton.html
(provided by the Furman University Mathematics Department)

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