Euler's Method: A Numerical Method for Solving Differential Equations

Given a differential equation (D.E.) of the form $\frac{dy}{dx} = f(x, y)$ and an initial condition (x_0, y_0), we would like to a find function $y = f(x)$ that satisfies the D.E. and passes through the point (x_0, y_0). In some cases, one can identify the general solution to the D.E. by inspection or using methods such as separation of variables and integration. But sometimes, either we don't have the analytic tools in our tool bag to solve the D.E. or an analytic solution is not possible. That's when we need a numerical method for solving the D.E. One such method is Euler's Method. It is based on using tangent lines to "piece together" an approximation to the particular solution to the D.E.

Given a starting point (x_0, y_0) and a derivative $\frac{dy}{dx}$. We can write the equation of the tangent line to the solution curve at the point (x_0, y_0) as follows:

$y - y_0 = m(x - x_0)$, where m is the slope of the tangent line, given by $\left.\frac{dy}{dx}\right|_{(x_0, y_0)}$.

If we solve for *y*, we get

$$y = y_0 + m(x - x_0), \text{ where } m = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}. \qquad (1)$$

Let's consider using this linear approximation for a "small" interval on the *x*-axis, say some Δx. Now consider the linear approximation at this new *x*-value, x_1, $x_1 = x_0 + \Delta x$. To find the *y*-value, y_1, at x_1, we substitute into equation (1), and we have

$$y_1 = y_0 + \left.\frac{dy}{dx}\right|_{(x_0, y_0)} (x_1 - x_0)$$

Now we have a new point (x_1, y_1). With this point we can write an equation for the line through the point (x_1, y_1) with the slope $\frac{dy}{dx}$ at (x_1, y_1), and we get

$$y - y_1 = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} (x - x_1) \qquad (2)$$

And we can find our new y-value, by substituting x_2, where $x_2 = x_1 + \Delta x$, into equation (2).

$y_2 = y_1 + \left.\frac{dy}{dx}\right|_{(x_1, y_1)} (x_2 - x_1)$. Replacing $x_2 - x_1$ with Δx, we have

$$y_2 = y_1 + \left.\frac{dy}{dx}\right|_{(x_1, y_1)} \Delta x$$

Continuing this process we can see that

$y_3 = y_2 + \left.\frac{dy}{dx}\right|_{(x_2, y_2)} \Delta x$ and

in general

$$x_n = x_{n-1} + \Delta x$$

$$y_n = y_{n-1} + \left.\frac{dy}{dx}\right|_{(x_{n-1}, y_{n-1})} \Delta x$$

We have a recursive set of equations that will generate a set of points that approximates the particular solution to the D.E.

From the recursive equation for generating the *y*-values, you can see that we get our new *y*-value by taking the "old" *y* value and adding to it the change in *y* times delta *x*. Here's an example of how we can use the method to generate numerical Euler's Method values:

Given the D.E. $\frac{dy}{dx} = -\frac{x}{y}$, with initial condition $(1, 4)$ and a given $\Delta x = 0.5$, we have

$$\begin{aligned} x_1 &= 1 + 0.5 \\ x_2 &= 1.5 + 0.5 = 2 \\ x_3 &= 2 + 0.5 = 2.5 \end{aligned}$$

and

$$y_1 = 4 + \left.\frac{dy}{dx}\right|_{(1,4)} (0.5) = 4 - \frac{1}{4}(0.5) = 4 + 0.125 = 3.875$$

$$y_2 = 3.875 + \left.\frac{dy}{dx}\right|_{(1.5, 3.875)} (0.5)$$

$$= 3.875 - \frac{1.5}{3.875}(0.5) \approx 3.681$$

$$y_3 = 3.681 + \left.\frac{dy}{dx}\right|_{(2,3.681)} (0.5)$$

$$= 3.681 - \frac{2}{3.681}\left(0.5\right) \approx 3.410$$

So if we are interested in an approximation for the solution at $x = 3.5$, we would use this recursive set of equations and iterate 5 times to find y_5.

Of course our solution is more accurate if we use a smaller Δx.

Please refer to the Excel file “euler_intro.xls”. We can see that the approximation for the solution at $x = 3.5$ is 2.5503 for $\Delta x = 0.5$ and 2.2687 for $\Delta x = 0.1$. The analytic solution yields a value of $\sqrt{17 - 3.5^2} \approx 2.18$ at $x = 3.5$.