Quadratic Euler's Method: An Improvement to Euler's Method

Given a differential equation (D.E.) of the form $\frac{dy}{dx} = f(x, y)$ and an initial condition (x_0, y_0), we can use Euler's method to approximate values of y associated with each value of x.

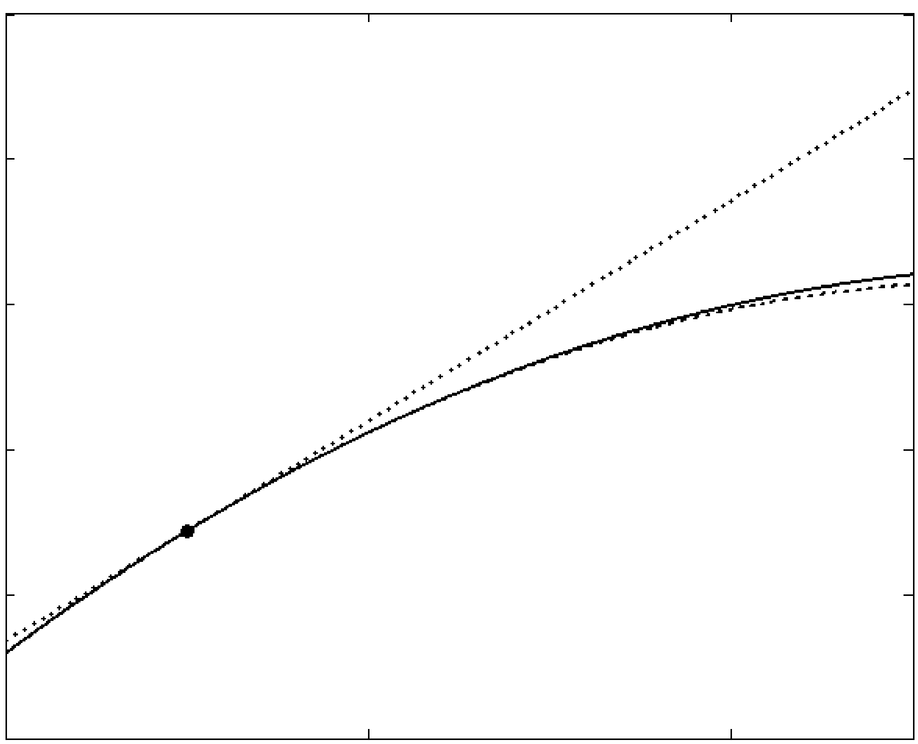
$$x_n = x_{n-1} + \Delta x$$

$$y_n = y_{n-1} + \frac{dy}{dx}(x_{n-1}, y_{n-1})\Delta x$$

By employing Euler's method we are following a succession of tangent line approximations starting at (x_0, y_0)each for an interval of Δx. However, the trouble with lines is that they don't curve! In general, the larger our Δx the larger our error in using Euler's method to predict y_n. Of course, we can reduce our step-size, Δx, in order to improve our approximation, but there are other ways to improve our accuracy without reducing our step-size.

Rather than using lines, which can only match the slope of the solution function at a given point, in order to approximate it, we can instead approximate by using a curve which matches both the slope and the concavity of the solution function at that point. That is, we can use a quadratic approximation (a.k.a. a "tangent parabola") to approximate by forcing the first and second derivatives of the quadratic to match the first and second derivatives of the function at the given point.

The illustration compares a both a linear approximation and quadratic approximation of a given function.



General Equations for Quadratic Euler's Method

The equation for the parabola tangent to $f(x)$ at $x = a$ is $y = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$. This is also called the Taylor polynomial of degree two for $f(x)$ about $x = a$.

Using this, we get the following equations for quadratic Euler's Method:

$$x_n = x_{n-1} + \Delta x$$

$$y_n = y_{n-1} + \frac{dy}{dx}(x_{n-1}, y_{n-1})\Delta x + \frac{1}{2} \cdot \frac{d^2y}{dx^2}(x_{n-1}, y_{n-1})(\Delta x)^2$$

Quadratic Euler's Method for the Flu Model

1. Using the differential equations found for the SIR model of the flu, write the quadratic Euler's method equations.

2. Use quadratic Euler's method to generate values and graphs of S, I, and R. Use the following initial values and constants: $N = 50,000$, $S_0 = 49,990$, $I_0 = 10$, $\alpha = 0.000005$, $\beta = \frac{1}{7}$, and $\Delta t = 1$.

3. Choose one of the other sets of values used in the original numerical investigation. Use quadratic Euler's method to generate values and graphs of S, I, and R.

4. How do the values/graphs obtained using quadratic Euler's method compare to those obtained using linear Euler's method?