Geometric Probability Example Solutions

The Parachute Problem

To avoid getting caught in a tree, the parachutist must land in the region shaded below:

\[
\text{Probability of not getting caught} = \frac{\text{area of shaded region}}{\text{area of field}}
\]

\[
\text{area of field} = 4 \, \text{km}^2
\]

\[
\text{area of shaded region} = 4 - \pi \left( \frac{1}{11} \right)^2 \, \text{km}^2 = 4 - \frac{\pi}{121} \, \text{km}^2
\]

\[
\text{desired probability} = 1 - \frac{\pi}{484} \approx 0.9935
\]

The Fairground Problem

If the distance from the center of the coin to the edge is less than or equal to the radius of the coin \((R)\), the coin will touch the edge of a square. So the center of the coin must be more than \(R\) units from each side in order for the player to win a prize. In the diagram below, the shaded region represents the event space:

\[
\text{area of square} = S^2
\]

\[
\text{area of shaded region} = (S - 2R)^2
\]

\[
\text{desired probability} = \frac{(S - 2R)^2}{S^2}
\]
The Triangle Problem

*Note: In this problem, we find the desired probability using a ratio of lengths rather than a ratio of areas.*

Let the line segment $AB$ represent the longer dowel and the line segment $CD$ represent shorter dowel.

If $L =$ length of $AB$, then the length of $CD = \frac{L}{2}$.

Let $X$ represent the random breaking spot on $AB$. The event space is the set of points $X$ for which $AX$, $XB$, and $CD$ form a triangle.

![Diagram of the triangle problem]

If $d =$ length of $AX$, then we know that a triangle will be formed if and only if:

- Length of $AX$ + Length of $XB$ > Length of $CD$: $d + (L - d) > \frac{L}{2} \Rightarrow L > \frac{L}{2}$ (trivially true)
- Length of $AX$ + Length of $CD > Length of XB$: $d + \frac{L}{2} > L - d \Rightarrow d > \frac{L}{4}$
- Length of $XB$ + Length of $CD > Length of AX$: $(L - d) + \frac{L}{2} > d \Rightarrow d < \frac{3L}{4}$

So, the event space will consist of all points $X$ that fall between a distance of $\frac{L}{4}$ and $\frac{3L}{4}$ from $A$.

The length of the event space is $\frac{L}{2}$ and the length of the sample space is $L$. So the desired probability is $\frac{1}{2}$.

The Tape Recorder Problem

1. Similar to the triangle problem, we will use ratios of lengths to find the probability.

Let $x =$ the starting time of the erasure.

Since the erasure is 45 minutes long, the sample space consists of all values of $x$ from 0 to 45.

We can represent the tape using a number line as shown below:

![Diagram of the tape recorder problem]

If the whole conversation is erased, the erasure can start as early as the 14th minute and as late as the 21st minute. So the event space is the set of all $x$ between 14 and 21.

\[
\begin{align*}
\text{length of sample space} & = 45 \\
\text{length of event space} & = 7 \\
\text{desired probability} & = \frac{7}{45} \approx 0.1556
\end{align*}
\]
2. Again we let $x =$ the starting time of the erasure.
If part of the conversation is erased, the erasure must start before the $29^{\text{th}}$ minute:  $x < 29$
And the erasure must end after the $21^{\text{st}}$ minute:  $x + 15 > 21 \Rightarrow x > 6$.
So the event space is the set of all $x$ between 6 and 29.

$$\text{length of sample space} = 45$$
$$\text{length of event space} = 23$$
$$\text{desired probability} = \frac{23}{45} \approx 0.5111$$

3. For this problem, we have two random events occurring: the starting time of the erasure and the starting
time of the conversation.

Again let $x =$ the starting time of the erasure. Also, let $y =$ the starting time of the conversation.

As in parts 1 & 2, the sample space consists of all values of $x$ between 0 and 45. Since the conversation is
8 minutes long and the erasure starts after the $21^{\text{st}}$ minute, the sample space consists of all values of $y$
between 21 and 52. The sample space can be represented by the shaded rectangle shown below:

For the entire conversation to be erased, we need the conversation:

- to begin after (or at the same time as) the erasure:  $y \geq x$
- to start by the time 7 minutes of the erasure have gone by:  $y \leq x + 7$

Thus the event space consists of all points $(x, y)$ that satisfy the inequality $x \leq y \leq x + 7$. This region is
shaded below:
Problems involving Pairs of Random Numbers

1. Suppose two numbers, $x$ and $y$, are generated at random, where $0 < x < 3$ and $0 < y < 6$. What is the probability that the sum is less than or equal to 2?

Need $x + y \leq 2 \Rightarrow y \leq -x + 2$.

$$\text{area of sample space} = (3)(6) = 18$$

$$\text{area of event space} = 2$$

$$\text{desired probability} = \frac{2}{18} \approx 0.111$$

2. Suppose two numbers, $x$ and $y$, are generated at random, where $0 < x < 1$ and $0 < y < 1$. What is the probability that the quotient $\frac{y}{x}$ is between 2 and 3?

Need $2 < \frac{y}{x} < 3 \Rightarrow 2x < y < 3x$.

$$\text{area of sample space} = 1$$

$$\text{area of event space} = \frac{1}{12}$$

$$\text{desired probability} = \frac{1}{12} \approx 0.0833$$
3. Suppose two numbers, $x$ and $y$, are generated at random, where $0 < x < 1$ and $0 < y < 1$. What is the probability that the product of the two numbers is less than $\frac{1}{2}$ (**requires Calculus to get exact value)

Need $xy < \frac{1}{2} \Rightarrow y < \frac{1}{2x}$.

area of sample space = 1

area of event space = $\frac{1}{2} - \frac{1}{2} \ln \left( \frac{1}{2} \right) \approx 0.8466$

desired probability $\approx 0.8466$

4. Suppose two numbers, $x$ and $y$, are generated at random, where $0 < x < 4$ and $0 < y < 4$. What’s the probability that the sum of the numbers exceeds the product?

Need $x + y > xy \Rightarrow xy - y < x \Rightarrow y(x - 1) < x$

For $0 < x < 1$, this gives $y < \frac{x}{x-1} = 1 + \frac{1}{x-1}$. For $1 < x < 4$, this gives $y > \frac{x}{1-x} = 1 + \frac{1}{1-x}$.

area of sample space = 16

area of event space $\approx 10.197$

desired probability $\approx 0.637$

Buffon’s Needle Problem

Let $y =$ distance from the lower line to the bottom of the needle and let $\theta =$ angle between the needle and the horizontal as shown below:

The sample space consists of all points $(\theta, y)$ for which $0 \leq \theta \leq \pi$ and $0 \leq y < D$. 
To determine the event space, let $V = \text{vertical distance from the bottom of the needle to the top of the needle.}$

Then we have $\sin \theta = \frac{V}{L} \Rightarrow V = L \sin \theta$.

The needle will lie across a line if $V > D - y \Rightarrow y > D - V = D - L \sin \theta$.

We need to separately consider the scenario for $L \leq D$ and $L > D$.

Suppose $L \leq D$. Then the event space would be the region shaded below:

- **area of sample space** = $\pi D$
- **area of event space (found using calculus)** = $2L$

\[
\text{desired probability} = \frac{2L}{\pi D}
\]

If $L > D$, we need to consider where the curve $y = D - L \sin \theta$ is zero. In this case,

\[
\text{desired probability} = \frac{\pi D - 2 \left(D \sin^{-1} \left(\frac{D}{L}\right) + \sqrt{L^2 - D^2} - L\right)}{\pi D}
\]

**Estimating $\pi$:**

If one were to do repeated trials of this experiment using values of $L$ and $D$ for which $L \leq D$, an experimental probability for the desired event could be obtained. Let’s call this probability $E$.

Then the quotient $\frac{2L}{D \cdot E}$ would give an approximation of $\pi$!