Guess My Function!

In each of the problems below, a function is specified by its properties. Use those properties to identify the function.

1. Let \( f \) be a function that is positive and differentiable for all \( x \). In addition, function \( f \) has the two properties:

i) \( f(a+b) = \frac{f(a)+f(b)}{f(-a)+f(-b)} \) for all \( a, b \in \mathbb{R} \) and ii) \( f'(0) = -1 \)

(a) Find the value of \( f(0) \).

Since \( f'(0) = -1 \), we know that \( f \) is continuous defined at zero. To find \( f(0) \) we can evaluate

\[
f(0+0) = \frac{f(0)+f(0)}{f(0)+f(0)} = 1.
\]

(b) Show that \( f(-x) = \frac{1}{f(x)} \) for all \( x \in \mathbb{R} \).

Find the values of \( f(x) \) and \( f(-x) \) and compare.

Now consider \( f(-x) = f(0+(-x)) = \frac{f(0)+f(-x)}{f(0)+f(x)} = \frac{1+f(-x)}{1+f(x)}. \)

Also, look at \( f(x) = f(0+x) = \frac{f(0)+f(x)}{f(0)+f(-x)} = \frac{1+f(x)}{1+f(-x)}. \) From these two computations we see that, indeed, \( f(-x) = \frac{1}{f(x)}. \).
(c) Show that \( f(x+h) = f(x) f(h) \) for all real numbers \( h \) and \( x \).

From the definition, \( f(x+h) = \frac{f(x) + f(h)}{f(-x) + f(-h)} \) and from (b), we have

\[
\frac{f(x) + f(h)}{f(-x) + f(-h)} = \frac{1}{f(x) + f(h)} \cdot \frac{f(x) + f(h)}{f(x) f(h)}.
\]

If we simplify the complex fraction, we find

\[
\frac{f(x) + f(h)}{f(x) f(h)} = f(x) f(h).
\]

(d) Use the definition of derivative to find \( f'(x) \) in terms of \( f(x) \).

First consider the difference quotient, then take the limit.

\[
\frac{f(x+h) - f(x)}{h} = \frac{f(x) f(h) - f(x)}{h} = f(x) \cdot \frac{f(h)-1}{h}.
\]

We need to know \( \lim_{h \to 0} \frac{f(h)-1}{h} \).

But \( \lim_{h \to 0} \frac{f(h)-1}{h} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = f'(0) \) is a rewriting of the definition of derivative.

We are given \( f'(0) = -1 \), so \( \lim_{h \to 0} f(x) \cdot \frac{f(h)-1}{h} = -f(x) \).

(e) What is function \( f \)?

We know that \( \frac{dy}{dx} = -y \) and \( f(0) = 1 \), so \( f(x) = e^{-x} \).
2. Let $g$ be a function that is differentiable throughout its domain and that has the following properties:

i) $g(x + y) = \frac{g(x) + g(y)}{1 - g(x)g(y)}$ for all real numbers $x, y$ and $x + y$ in the domain of $g$.

ii) $\lim_{h \to 0} g(h) = 0$ and iii) $\lim_{h \to 0} \frac{g(h)}{h} = 1$.

a) Show that $g(0) = 0$.

As before, consider $g(0 + 0) = \frac{g(0) + g(0)}{1 - g(0)g(0)}$, so $g(0) = \frac{2g(0)}{1 - g^2(0)}$. Solving for $g(0)$, we have $g(0)(1 - g^2(0)) = 2g(0)$. So, $g(0) = 0$ is one solution to the equation. Also consider $(1 - g^2(0)) = 2$, which has no solution. Then $g(0) = 0$.

b) Use the definition of derivative to show that $g'(x) = 1 + g^2(x)$

Using the definition, the difference quotient is

$$
\frac{g(x + h) - g(x)}{h} = \frac{\frac{g(x) + g(h)}{1 - g(x)g(h)} - g(x)}{h} = \frac{g(x) + g(h) - g(x) + g^2(x)g(h)}{h(1 - g(x)g(h))}.
$$

Simplifying this expression gives $\frac{g(h)}{h} \cdot \frac{(1 + g^2(x))}{(1 - g(x)g(h))}$. We are given that $\lim_{h \to 0} g(h) = 0$ and $\lim_{h \to 0} \frac{g(h)}{h} = 1$, so

$$
\lim_{h \to 0} \frac{g(h)}{h} \cdot \frac{(1 + g^2(x))}{(1 - g(x)g(h))} = (1) \frac{(1 + g^2(x))}{(1 - g(x)g(x)(0))} = 1 + g^2(x).
$$

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\lim_{h \to 0} \frac{g(h)}{h} \cdot \frac{(1 + g^2(x))}{(1 - g(x)g(h))} = (1) \frac{(1 + g^2(x))}{(1 - g(x)g(x)(0))} = 1 + g^2(x).
$$

c) Use b) to find $g(x)$.

Given $\frac{dy}{dx} = 1 + y^2$, we have $\int \frac{dy}{1 + y^2} = \int dx$. Integrating we find that $\tan^{-1}(y) = x + c$. With initial condition $g(0) = 0$, we see that $c = 0$. Then $\tan^{-1}(y) = x$ and $y = \tan(x)$ with $\frac{-\pi}{2} < x < \frac{\pi}{2}$.