Predicting the Spread of an Infectious Disease

The world is being threatened by a new disease, SARS. The Center for Disease Control (CDC) offers the following information at [http://www.cdc.gov/ncidod/sars/factsheet.htm](http://www.cdc.gov/ncidod/sars/factsheet.htm):

A new disease called SARS: Severe acute respiratory syndrome (SARS) is a respiratory illness that has recently been reported in Asia, North America, and Europe. This fact sheet provides basic information about the disease and what is being done to combat its spread. To find out more about SARS, go to [www.cdc.gov/ncidod/sars/](http://www.cdc.gov/ncidod/sars/) and [www.who.int/csr/sars/en/](http://www.who.int/csr/sars/en/). The Web sites are updated daily.

Symptoms of SARS: In general, SARS begins with a fever greater than 100.4°F [>38.0°C]. Other symptoms may include headache, an overall feeling of discomfort, and body aches. Some people also experience mild respiratory symptoms. After 2 to 7 days, SARS patients may develop a dry cough and have trouble breathing.

How SARS spreads: The primary way that SARS appears to spread is by close person-to-person contact. Most cases of SARS have involved people who cared for or lived with someone with SARS, or had direct contact with infectious material (for example, respiratory secretions) from a person who has SARS. Potential ways in which SARS can be spread include touching the skin of other people or objects that are contaminated with infectious droplets and then touching your eye(s), nose, or mouth. This can happen when someone who is sick with SARS coughs or sneezes droplets onto themselves, other people, or nearby surfaces. It also is possible that SARS can be spread more broadly through the air or by other ways that are currently not known.

Possible cause of SARS: Scientists at CDC and other laboratories have detected a previously unrecognized coronavirus in patients with SARS. The new coronavirus is the leading hypothesis for the cause of SARS.

In this investigation, we will develop a model for the spread of SARS in a closed population. The model is similar to those used to model the spread of other infectious diseases like measles, influenza, and AIDS. We are particularly interested in considering the effectiveness of isolation and quarantine of infected individuals, and wearing of masks by individuals concerned about “catching” SARS. We will consider what happens when a small group of people with an infectious disease is introduced into a larger, closed population? Does everyone in the larger population get the disease. Will the disease die out before it spreads? Is there some critical mass of infectious people that must exist for the disease to become an epidemic? When will the disease reach its maximum number, and what fraction of the total population has the disease at that time?

The Basic Model

The standard model for the spread of infectious diseases like SARS is known as a compartment model, since we think of people moving from one compartment to another. We assume we have a fixed population of $N$ individuals through which an infectious disease is moving. Some of the people have the disease and are called Infectives. Some of the people do not yet have the disease but may catch it if they interact with an Infective. These are called Susceptibles. Some of the people may have already had the disease and have recovered from it. They are called Recovereds. For some diseases, the Recovereds develop an immunity to the disease, while for others they return to the Susceptible group and can again come down with the
disease. For other diseases, there is no recovery. All cases lead to death, so the Recovereds category contains only those who have died due to the disease.

![Compartment Model for SARS](image)

**Figure 1: Compartment Model for SARS**

**Modeling the Transition**

We will assume that the disease has a very short incubation period, so that immediately after contacting the disease, the infected person can pass it on (there is a 10 day period in which the infected individual does not have symptoms, but can transmit the disease). We are also assuming in our model that once an individual recovers from the disease, that individual becomes permanently immune (or the disease results in death), and cannot catch it again. It is not known at present if recovery gives immunity from SARS. These assumptions allow us to divide the population into four groups:

- the infected group \((I)\) consisting of those individuals who presently have the disease and can transmit it to others. These are called *Infectives*.
- the susceptible group \((S)\) consisting of those individuals who are not infected, but are capable of becoming infected. These are called *Susceptibles*.
- the recovered group \((R)\) consisting of those who have had the disease and recovered and cannot become infected again. These are called *Recovereds*.
- the terminal group that has died as a result of the disease \((T)\). These are called *Terminals*.

**The spread of the disease is governed by four rules:**

1. The base population remains constant throughout the duration of the disease. We will neglect births, deaths due to factors unrelated to the disease, and immigration into or emigration out of the base population.

2. For most infectious diseases, transmission happens when an Infective comes in “contact” with a Susceptible. This contact could be physical contact as in many STD’s, or contact via a cough or door handle, or bites from the same mosquito. Not all Infectives interact will all Susceptibles, but the larger the sub-population of Susceptibles, the greater the probability of an interaction. Likewise, the larger the sub-population of Infectives, the greater the probability of an interaction. Since not all contacts result in transmission of the disease, we can describe the rate of transmission as \(k \cdot S \cdot I\), where the value of \(k\) carries with it both the probability of interaction between the two and the probability of transmission given an interaction.

A large value of \(k\) suggests a disease that is easily transmitted through personal interaction, while small values of \(k\) suggest a disease that is difficult to transmit through personal interaction. SARS is passed when an infected individual comes into contact with a susceptible individual. Not every contact results in transmission of the disease (thank goodness). The
product of $S$ and $I$ is a measure of how many possible contacts there are (this is similar to the predator-prey setting). There is a 10 day incubation period before symptoms show.

3. Individuals are removed from the infectious group at a rate proportional to the size of $I$. Call the constant of proportionality $r$. The value of the parameter $r$ measures the rate of recovery (or death) from the disease. If a disease lasts a long time, $r$ is small. If the disease is rapid, $r$ is large.

4. Some of the individuals removed from the infectious group become Recovereds while others die from the disease and becomes Terminals. At the present time, the death rate from SARS is approximately 15%.

**Computer (or Calculator) Investigation**

Before doing any analysis, we want to set up and “play” with the model. We will use Euler's method to investigate how the values of $k$, $r$, $S_0$, and $I_0$ affect the dynamics of the system. Start with a simple system using total population $N = 50,000$, $S_0 = 49,990$, $R_0 = 10$, $I_0 = T_0 = 0$. This is our initial population for the investigation. We will later modify these values to see how the progress of the epidemic is altered by the change. At the present time, the research community has no good value for $k$. For our investigation, we will assume initially that $k = 0.000005$, a value that is half of the rate often used for the spread of measles.

1. Write a system of 4 coupled differential equations describing $\frac{dS}{dt}$, $\frac{dI}{dt}$, $\frac{dR}{dt}$, and $\frac{dT}{dt}$ in terms of $I$, $S$, $R$, $T$, $k$, and $r$.

2. Set up the Euler iteration for this system of coupled differential equations. Start with a simple system using $N = 50,000$, $S_0 = 49,990$, $I_0 = 10$, $R_0 = D_0 = 0$, and $k = 0.000005$. Use $\Delta t = 1$ day for your iteration.

3. From Table 1, determine an approximate value for $r$ in your investigation. Be sure to include the 10 day incubation period before symptoms occur. Your working iteration should look similar to the graph below:

![Graph of disease progression](image)

4. How does the progress of the disease change if $N = 40,000$, $N = 20,000$ or $N = 10,000$. For each of these populations, estimate the total number who eventually come down with SARS and the maximum number ill with SARS at one time.
5. With $N = 50,000$, suppose the disease is more difficult to catch than our model suggests. Use $k = 0.0000025$. How does this affect the total number who come down with SARS and the maximum number ill with SARS at one time? Suppose the SARS virus is as easy to catch as measles, how does this affect the total number who come down with SARS and the maximum number ill with SARS at one time?

6. At the present time, patients are treated with antibiotics, but no one knows if this treatment is effective. Using $N = 50,000$ and $k = 0.000005$ again, suppose a treatment is found that cuts the recovery time in half. How does this affect the total number who come down with SARS and the maximum number ill with SARS at one time?

7. Many people in areas where SARS is widespread try to reduce the spread of the disease by wearing masks. The Canadian Broadcast Corporation (CBC) did a report of the ability of masks to reduce the number of microscopic particles breathed in. Table 2 has some results of their study. Using $N = 50,000$ and $k = 0.000005$, if the value of $k$ is reduced by the suggested percentages, how would the progression of the disease be affected. If everyone wore Dust Masks, would that appreciably affect the disease? Suppose everyone wore Surgical Masks? N-95 masks?

**Analytic Investigation**

In our investigation using Euler’s Method, we have developed a number of conjectures about what would happen to the progress of the disease under different conditions. We want to verify those results and learn some crucial facts about epidemics by using Calculus.

1. When is $I(t)$ increasing and when is it decreasing? What does this mean about the spread of the disease. Using this result, explain why quarantine will be effective against SARS.

2. We have equations for $\frac{dI}{dt}$ and $\frac{dS}{dt}$. Use them to find $\frac{dI}{dS}$. Solve this differential equation for $I(S)$. Find $I'(S)$ and $I''(S)$ and use these derivatives to determine when the number of infected will begin to decrease. Does this match your solution from 1)? Explain any differences you see.

3. Use the function $I(S)$ to determine the maximum number of individual ill at one time. What proportion of the population will be sick simultaneously? Are these results consistent with your graphs in the computer investigation?

4. How many Infecteds will there be when the epidemic is over? Use this idea to determine the proportion of the population infected under the different situations considered in the computer investigation. How effective do the masks need to be to seriously affect the progress of the disease? Is it realistic to think isolation and wearing masks can work or must a vaccine be found?
Table 1: Severe Acute Respiratory Syndrome (SARS) in Singapore: Clinical Features of Index Patient and Initial Contacts

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This table presents information about the first 19 SARS patients in China. The numbers in the table are days after the onset of symptoms.

<table>
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<th>Patient</th>
<th>Sex</th>
<th>Age</th>
<th>Onset date</th>
<th>Hospital admission</th>
<th>1st abnormal chest x-ray</th>
<th>Supplemental oxygen requirement</th>
<th>Mechanical ventilation</th>
<th>Fever resolved</th>
<th>Radiologic improvement</th>
<th>(Death) medically fit for discharge</th>
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<td>4</td>
<td>5</td>
<td>9</td>
<td>10</td>
<td>Hosp</td>
</tr>
</tbody>
</table>

*Abbreviations used in table: F, female; M, male; NR, not required; NS, not seen; Hosp, hospitalized.

bIndex patient.

Ahead of Print

Vol. 9, No. 6
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**Table 2: CBC News**

Air Date: Apr 8, 2003  
Reporter: Wendy Mesley  
Producer: Ines Colabrese  
Researcher: Marlene McArdle, Colman Jones, Louisa Jaslow

Ugis Bickis is an Environmental Hygienist who teaches at Queens University in Kingston. He’s been studying the effectiveness of masks for years. But he found a little surprise when he went shopping recently.

We bought N95s, the ones recommended for health care workers. Then we randomly picked a procedure mask (the kind you might see on your dentist), a surgical mask (like you’d see in a hospital) and one of those dust masks you often find in hardware stores, used for painting and other messy work. We also tossed in a bandana — a sampling of the products some Canadians have been using since the SARS scare began.

The test result is the “Filtration Efficiency”, it’s the percentage of sub micron particles that the mask filtered out. The higher the number, the greater number of particles are filtered out.

These values are averages from the article:

- **Bandanna:** Folded Twice: 6 per cent of particles filtered out, Folded 4 times: 10 per cent  
- **Filter Mask:** 12 per cent  
- **Dust Mask:** 13 per cent  
- **Procedure Mask:** 32 per cent  
- **Surgical Mask:** 62 per cent  
- **N-95:** 98 per cent
Sample SARS Solution:

If we let $I(t)$ represent the number of people in the population that are infected at time $t$, $S(t)$ represent the number of people in the population that are susceptible at time $t$, $R(t)$ represent the number of people in the population that are recovered at time $t$, and $T(t)$ represent the number of people in the population that have died from the disease, then the system of equations that represents the spread of an infectious disease is:

Modeling the Transition Rate

For most infectious diseases, transmission happens when an Infective comes in “contact” with a Susceptible. This contact could be physical contact as in many STD’s, or contact via a cough or door handle, or bites from the same mosquito. Not all Infectives interact will all Susceptibles, but the larger the sub-population of Susceptibles, the greater the probability of an interaction. Likewise, the larger the sub-population of Infectives, the greater the probability of an interaction. Since not all contacts result in transmission of the disease, we can describe the rate of transmission as $\alpha \cdot S \cdot I$, where the value of $\alpha$ carries with it both the probability of interaction between the two and the probability of transmission given an interaction.

There is no Gain for Susceptibles, no Loss for Recovereds, and the Gain for Infectives is the Loss for Susceptibles while the Loss for Infectives is the Gain for Recovereds. The only real change is to add a rate of recovery, $\beta$. During each time interval, a proportion of those infected will recover. They can no longer spread the disease and are immune to catching it again. For many diseases, if the infected have the disease for k days, the recovery rate is estimated as $\beta = \frac{1}{k}$.

Unfortunately, for some diseases, “recovery” may mean death due to the disease.

\[
\begin{align*}
\frac{dS}{dt} &= -k S \cdot I \\
\frac{dI}{dt} &= k S \cdot I - r I \\
\frac{dR}{dt} &= (1-a)(r I) \\
\frac{dT}{dt} &= a( r I) \\
\text{with } S(t) + I(t) + R(t) + T(t) &= N
\end{align*}
\]

The parameters $k$, $r$, and $a$ represent the transmission rate, recovery rate, and death rate from the disease, respectively. At the present time the value of $a$ (death rate) is estimated to be 0.15. A simpler option is to consider

\[
\begin{align*}
\frac{dS}{dt} &= -k S \cdot I \\
\frac{dI}{dt} &= k S \cdot I - r I \\
\frac{dR}{dt} &= r \cdot I \\
T(t) &= a \cdot R \\
\text{with } S(t) + I(t) + R(t) &= N
\end{align*}
\]

This model puts together the Recovereds and Terminals, then pulls out the proportion that die from the disease. The count of Recovereds includes all Terminals.
Euler’s Method Investigation

For our initial Euler’s exploration, we will use $N = 50,000, S_0 = 49,990, I_0 = 10, I_0 = T_0 = 0, k = 0.000005, \text{ and } a = 0.15$. We need to find an appropriate value for $r$. The mean hospital stay is actually 18.6, but two patients on the list were not yet released from the hospital, so the mean time could easily be longer. I will use an average time from hospitalization until death or recovery of 20 days for the investigation. Since there is a 10 day incubation period before symptoms appear, we can say that the average time from infection to recovery or death is 30 days. So, one-thirtieth of those infected should recover each day. We will use $r = \frac{1}{30} \approx 0.0333$.

Now, we can write our Euler’s Method iterations. $t_{n+1} = t_n + \Delta t$

\[
S_{n+1} = S_n - (k \cdot S_n \cdot I_n) \Delta t \\
I_{n+1} = I_n + (k \cdot S_n \cdot I_n - r \cdot I_n) \Delta t \\
R_{n+1} = R_n + \left( (1 - a) r \cdot I_n \right) \Delta t \\
T_{n+1} = T_n + (a \cdot r \cdot I_n) \Delta t
\]

Since $I_0 = 10$ in each case, when $N = 50,000$ then $S_0 = 49990$. The graphs below illustrate the development of the disease with different populations. Euler’s Method is used to approximate the total number developing the disease. The table below gives the number developing the disease, the number of expected deaths, and the number and proportion ill at the height of the outbreak.

### Changing the Number of Susceptibles

![Graphs illustrating Euler’s Method iterations for different populations.](image)

- **N = 50,000**
- **N = 40,000**
- **N = 30,000**
- **N = 20,000**
Population | Total Sick | Expected Deaths | Maximum Sick | Max. Proportion |
--- | --- | --- | --- | --- |
50,000 | 47,482 | 7,122 | 31,000 | 62% |
40,000 | 37,923 | 5,688 | 22,500 | 56% |
30,000 | 28,200 | 4,230 | 13,500 | 45% |
20,000 | 17,908 | 2,686 | 6,200 | 31% |
10,000 | 5,557 | 834 | 600 | 6% |
5,000 | 37 | 6 | 10 | 0.2% |

There is a gradual decline in the total number ill with SARS until $N < 20,000$, at which point there is a dramatic decrease. If $N = 10,000$ only about half of the population contracts SARS while if $N = 5,000$, less than eventually 1% become ill. Clearly, the size of $N$ is important in the development of the epidemic, but in what way?

**Changing the Rate of Transmission**

If the disease is more difficult to transmit than our first model suggests ($k = 0.0000025$), the ultimate number of people who come down with the disease is unchanged at around 46,300, but the maximum number at one time is reduced to around 19,500. The disease takes longer, but essentially, everyone is still infected.

If both the population size and transmission rate are changed ($k = 0.0000025$ and $N = 30,000$) there is a dramatic effect on the course of the disease.
If the disease is more easily transmitted, the progress of the disease is very rapid. If \( k = 0.000001 \), everyone becomes infected in a very short period of time (35 days). If \( k = 0.0000001 \), only 28,000 of the 50,000 Susceptibles become infected. Reducing the transmission rate is a major goal for health officials.

**Changing the Rate of Recovery**

The rate of recovery in the initial simulation was \( r = 0.0333 \). If we cut that in half with more effective treatment, the spread of the disease is affected only marginally. The maximum number ill is reduced, but 46,000 of the 50,000 Susceptibles eventually become ill. Compare the two graphs below.

At the present time, there is little that can be done about the 10 day incubation period before symptoms appear. If we are able to cure the disease as soon as symptom appear, but the victims are able to transmit the disease as soon as they contract if, then the minimum value of \( r \) is \( r = \frac{1}{10} \).
If it takes 5 days for the disease to develop before someone can transmit it to someone else, then we would have \( r = \frac{1}{5} \). The two graphs below illustrate these two situations.

If \( r = \frac{1}{10} \), then we expect around 43,000 to develop the disease and 6,450 deaths. If \( r = \frac{1}{5} \), then we expect only 18,000 to become infected resulting in 2700 deaths. The recovery rate is important, but there may be nothing that can be done to affect it. The transmission rate and size of the susceptible population are the most easily altered.

**Wearing Masks**

The effectiveness of masks can be compared based on their ability to remove particles from the air. It seems reasonable to expect that the transmission rate will be affected by a factor proportional to the amount of particulate stopped by the mask. For the sake of our model, if we assume that a 20% reduction in particles carrying the disease results in a 20% reduction in the rate of transmission (which may or may not be true, further study needs to be done), then we can compare the results.
Clearly, there is a dramatic dropoff in the spread of the disease between the 62% reduction from surgical masks and the 98% by N-95 masks.

<table>
<thead>
<tr>
<th>Mask Type</th>
<th>% reduction</th>
<th>Population</th>
<th>% Sick</th>
<th>Expected Deaths</th>
<th>Max. Proportion</th>
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<td>50,000</td>
<td>100%</td>
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<td>99%</td>
<td>2497</td>
<td>58%</td>
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Wearing the N-95 masks could have a dramatic effect on the spread of the disease. How practical is it to supply 50,000 N-95 masks in a short time?

**Analytic Investigation**

From the differential equation \( \frac{dI}{dt} = kSI - rI \), we know that \( \frac{dI}{dt} = I(kS - r) > 0 \) whenever \( S > \frac{r}{k} \). This is the important fact of the growth of the epidemic. In our first computer investigation, we found thought that \( N \) was important, but now we know it is really \( S \). The key to changing the dynamics of the disease, either by changing \( S_0, k \) or \( r \) was to get \( S < \frac{r}{k} \). If you look back at the graphs and numerical values, you will see that \( S = \frac{r}{k} \) is a critical value. In our initial problem with \( k = 0.000005 \) and \( r = 0.03333 \), \( \frac{r}{k} = 6667 \). Once the number of Susceptibles falls below 6,667, the epidemic dies.
To find the function determining the number of Infectives, we can eliminate $t$ from the equations by solving for $\frac{dI}{dS}$. We know that

$$\frac{dI}{dS} = \frac{kSI - rI}{-kSI} = -1 + \frac{r}{kS}.$$ 

Solving for $I(S)$, we have $\int dI = \int -1 + \frac{r}{kS} \, dS$, which simplifies to

$$I(S) = -S + \frac{r}{k} \ln(S) + C.$$ 

If $t = 0$, we have $S_0$ and $I_0$ Susceptibles and Infectives, respectively, and we know that

$$I(S) = S_0 + I_0 - S + \frac{r}{k} \ln \left( \frac{S}{S_0} \right).$$

We will use this function to consider answer the questions posed during the Euler’s Method investigation.

What does this function look like? We already know that $I'(S) = -1 + \frac{r}{kS}$. Since $I''(S) = -\frac{r}{k} \frac{1}{S^2}$ is always negative, we know that the function $I(S)$ is always concave down and has its maximum value at $S = \frac{r}{k}$. If $I' > 0$ the number of Infectives increases and if $I' < 0$, the number of Infectives is decreasing.

The graph of $I(S) = S_0 + I_0 - S + \frac{r}{k} \ln \left( \frac{S}{S_0} \right)$ can be deceiving since $S$ is always decreasing. So, when $S < \frac{r}{k}$, the epidemic will begin to wind down. As long as $S > \frac{r}{k}$, the epidemic will continue to build. The value $\frac{r}{k}$ is the ratio of the rate at which Infectives recover and Susceptibles become infected. Notice that if $S = \frac{r}{k}$, then both $\frac{dI}{dS}$ and $\frac{dI}{dt}$ are equal to zero. So we see that the ratio of $r$ to $k$ is important in the spread of the disease. The best way to restrict the spread of the epidemic is to affect this ratio.
If there is a large enough population of Susceptibles, $S > \frac{r}{k}$, then the number of infected individuals will increase. There must be a sufficient number of Susceptibles available for the epidemic to develop. This is why separating the infected from S Susceptibles by quarantine is important in halting the disease. Notice that the initial number of Infectives does not seem to matter, since it does not appear in the derivative. The epidemic will end, naturally (that is, without intervention), when the number of available Susceptibles is too small. This does not mean that everyone will eventually get the disease. Once $S = \frac{r}{k}$, the epidemic will begin to wind down. By isolating Infectives, we are effectively reducing the number of available Susceptibles.

With the equation $I(S) = S_0 + I_0 - S + \frac{r}{k} \ln \left( \frac{S}{S_0} \right)$, students should be able to give some analytical solutions to the investigations using Euler’s Method. For example, the initial problem setting gives $I(S) = 49990 + 10 - S + 6667 \ln \left( \frac{S}{49990} \right)$ with $I'(S) = -1 + \frac{6667}{S}$. So the epidemic should begin to wind down when $S = 6667$. Unfortunately, that’s after 43,333 have gotten the disease. If we reduce the value of $k$ by wearing masks to $k = (0.25)(0.000005) = 0.00000125$, then the epidemic will stall when $S = 26,667$.

**Maximum Number Ill**

What proportion of the population will have the disease at its peak. This affects the public’s perception of the seriousness of the outbreak. We can use our equation $I(S)$ and evaluate it at $S = \frac{r}{k}$? So

$$I \left( \frac{r}{k} \right) = S_0 + I_0 - \frac{r}{k} + \frac{r}{k} \ln \left( \frac{r/k}{S_0} \right).$$

In our example above, we had $\frac{r}{k} = 6667$, $S_0 = 49990$, and $I_0 = 10$, so the maximum number infected will be

$$I(6667) = 50000 - 6667 + 6667 \ln \left( \frac{6667}{49990} \right) = 29901$$

or almost 60% of the population sick at one time. If, however, we have a situation in which $\frac{r}{k} = 30,000$ (by wearing masks, for example), then we would have

$$I(30000) = 50000 - 30000 + 30000 \ln \left( \frac{30000}{49990} \right) = 4681$$

or 9% of the population. This is still a lot, but the size of the epidemic is dramatically reduced.

The maximum percentage of the population ill at one time contributes to the public’s perception of the scope of the disease and to the fear and panic that often accompanies an epidemic.
Total Ill and Total Deaths

The total number infected can be found using our function \( I(S) = S_0 + I_0 - S + \frac{r}{k} \ln \left( \frac{S}{S_0} \right) \).

The epidemic is over when \( I = 0 \). So, for each situation, we can use numerical methods to solve
\[
S_0 + I_0 - S + \frac{r}{k} \ln \left( \frac{S}{S_0} \right) = 0.
\]
In our initial example, \( \frac{r}{k} = \frac{1}{30} = 6667 \), then
\[
50000 - S + 6667 \ln \left( \frac{S}{49990} \right) = 0 \text{ at } S = 50000
\]
and everyone will eventually contract the disease. Since 15% die from the disease, there will be about 7500 deaths from SARS.

If we increase \( \frac{r}{k} \) to 30000 (by wearing masks, for example) we have
\[
50000 - S + 30000 \ln \left( \frac{S}{49990} \right) = 0 \text{ at around } S = 16205.
\]
This means 16,205 individuals did not become infected. Since 33,795 were infected, we expect to see 5069 deaths. This is an improvement. However, if we can increase \( \frac{r}{k} \) to 50000 the zero is at \( S = 49006 \), so only 994 would be expected to become ill and only 149 deaths. We are assuming that the reduction in transmission rate is proportional to the reduction in the proportion of particles passing through the mask. This seems reasonable, but using a constant of proportionality of 1 is probably overly optimistic. In a population of 50,000 people, if \( \frac{r}{k} > 50000 \) no outbreak will occur. With \( r = \frac{1}{30} \), if we can get \( k < 0.0000007 \) there will be no outbreak. Further study of the spread of the disease may give us enough information to determine how practicable such a reduction is and how to accomplish it.

References:
